YANGIANS, QUANTUM LOOP ALGEBRAS AND ABELIAN DIFFERENCE EQUATIONS

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To Andrei Zelevinsky (1953–2013). Advisor, mentor, colleague and friend.

ABSTRACT. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, and $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ the Yangian and quantum loop algebra of $\mathfrak{g}$. Assuming that $\hbar$ is not a rational number and that $q = e^{\pi i \hbar}$, we construct an equivalence between the finite–dimensional representations of $U_q(L\mathfrak{g})$ and an explicit subcategory of those of $Y_\hbar(\mathfrak{g})$ defined by choosing a branch of the logarithm. This equivalence is governed by the monodromy of the abelian, additive difference equations defined by the commuting fields of $Y_\hbar(\mathfrak{g})$. Our results are compatible with q–characters, and apply more generally to a symmetrisable Kac–Moody algebra $\mathfrak{g}$, in particular to affine Yangians and quantum toroïdal algebras. In this generality, they yield an equivalence between the representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ whose restriction to $\mathfrak{g}$ and $U_q\mathfrak{g}$ respectively are integrable and in category $\mathcal{O}$.

CONTENTS

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, and $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ the corresponding Yangian and quantum loop algebra. Recall that the latter are deformations of the enveloping algebras of the current Lie algebra $\mathfrak{g}[s]$ of $\mathfrak{g}$ and its loop algebra $\mathfrak{g}[z, z^{-1}]$ respectively.

These affine quantum groups have been known to be closely related since their introduction by Drinfeld in the mid 80s [14, 15]. In particular

(1) Drinfeld proved that, just as the loop algebra $\mathfrak{g}[z, z^{-1}]$ degenerates to the current algebra $\mathfrak{g}[s]$ by setting $z = \exp(\epsilon s)$ and letting $\epsilon \to 0$, the quantum loop algebra degenerates to the Yangian. Specifically, $Y_\hbar(\mathfrak{g})$ is the graded algebra associated to the filtration of $U_q(L\mathfrak{g})$ by powers of the ideal of $z = 1$ [15, 30].

(2) The highest weight classification of irreducible, finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ due to Drinfeld and Chari–Pressley respectively [16, 10] yields a surjection from the parameter space of finite–dimensional irreducible representations of $Y_\hbar(\mathfrak{g})$ to the corresponding parameter space for $U_q(L\mathfrak{g})$. This (set–theoretic) map $\mathcal{E}xp$ is given by exponentiating the roots of Drinfeld polynomials.

(3) Stronger results can be obtained when $\mathfrak{g}$ is of type ADE, by using Nakajima quiver varieties, and the geometric action of $U_q(L\mathfrak{g})$ on their equivariant K–theory [44]. The analogous action of the Yangian $Y_\hbar(\mathfrak{g})$ in cohomology due to Varagnolo [52] allows to relate irreducible representations of $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ via the Chern character, and shows in particular that the map $\mathcal{E}xp$ preserves dimensions.

1.2.   It has long been believed that $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ have the same representation theory (see, *e.g.,* [52, p. 283]). This is not directly suggested by (1) (there is after all no relation between the representations of a filtered algebra and of its associated graded in general) but is borne out by (2), (3) and the fact that many representations of $Y_\hbar(\mathfrak{g})$, for example evaluation representations of $Y_\hbar(\mathfrak{sl}_n)$, can be explicitly deformed to representations of $U_q(L\mathfrak{g})$.

This belief is further corroborated by the fact that, in the analogous case of the degenerate and affine Hecke algebras $\mathfrak{H}, \mathbb{H}$ of a Weyl group, a complete understanding of the relation between finite–dimensional representations of $\mathfrak{H}$ and $\mathbb{H}$ was obtained by Lusztig [40], and can be recovered by realising $\mathfrak{H}, \mathbb{H}$ as the equivariant cohomology and K–theory of the corresponding Steinberg variety [29].

Despite the above evidence, however, no precise, functorial relation between the categories $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ seems to have been known so far, even conjecturally. [1]

One notable exception is the works [35, 18, 19] which relate the representations of $U_q(L\mathfrak{sl}_n)$ and Felder's elliptic quantum group $E_{\tau,\gamma}(\mathfrak{gl}_n)$ with $q = e^{2\pi i\gamma}$, and construct in particular a fully faithful functor from $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{sl}_n))$ to $\mathrm{Rep}_{\mathrm{fd}}(E_{\tau,\gamma}(\mathfrak{gl}_n))$ [18]. A similar construction should give rise to a functor $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{sl}_n)) \to \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{gl}_n))$. These constructions depend, however,

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[1]Note that the Chern character in the geometric realisations (3) above cannot be considered a functor, since it is not known how to realise arbitrary finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ of $U_q(L\mathfrak{g})$ inside the equivariant cohomology or K–theory of a quiver variety.

on the RTT presentation of $U_q(L\mathfrak{sl}_n)$, which is less convenient for other classical Lie algebras, and currently unavailable for the exceptional ones.

1.3.   Our initial interest in this question stemmed from the monodromy conjecture formulated in [50], which relies implicitly on the existence of a functor between finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$. The conjecture states that the monodromy of the trigonometric Casimir connection, which yields representations of the affine braid group of $\mathfrak{g}$ on finite–dimensional $Y_\hbar(\mathfrak{g})$–modules, is described the quantum Weyl group operators of $U_q(L\mathfrak{g})$.

Motivated by this, we formulated the hope, naïvely suggested by (2) above, that $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ might be an exponential cover of $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, that is, that $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ is equivalent to a subcategory of $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ obtained by restricting the roots of Drinfeld polynomials to a domain $\Pi \subset \mathbb{C}$ mapping bijectively to $\mathbb{C}^\times$ via the exponential map [49].

1.4.   The goal of this paper is to prove that this is indeed the case.

In our previous work [27], we proved that, when the deformation parameters $\hbar, q$ are formal, and related by $q = e^\hbar$, the quantum loop algebra and Yangian are *isomorphic* after appropriate completions, specifically with respect to the natural loop grading on $Y_\hbar(\mathfrak{g})$ and the powers of the ideal of $z = 1$ in $U_q(L\mathfrak{g})$. The corresponding isomorphism

$$\Phi : \widehat{U_q(L\mathfrak{g})} \longrightarrow \widehat{Y_\hbar(\mathfrak{g})}$$

constructed in [27] deforms the change of variables $\mathfrak{g}[z, z^{-1}] \to \mathfrak{g}[[s]]$ given by $z = e^s$, and exponentiates the roots of Drinfeld polynomials. This strenghtened Drinfeld's degeneration result alluded to in 1.1, clarified the relation between $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, and suggested that their finite–dimensional representations be related by pull–back via Φ.

The framework of [27] is, however, not suited to the study of finite–dimensional representations since the map Φ is given by formal infinite series which do not converge on finite–dimensional representations and do not allow for the numerical specialisation of the deformation parameters.

1.5.   In this paper, we consider the case when $\hbar \in \mathbb{C}$ is not a rational number, so that $q = \exp(\pi i\hbar) \in \mathbb{C}^\times$ is not a root of unity. We shall work more generally with an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$ and, accordingly, consider the categories $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ consisting of representations whose restriction to $\mathfrak{g}$ and $U_q\mathfrak{g}$ respectively is integrable and in category $\mathcal{O}$ (see [32, §4] and §3.1). When $\mathfrak{g}$ is semisimple, $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ coincide with the categories of finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$.

1.6.   Our first main result is the following

Theorem. *If $\hbar \in \mathbb{C} \setminus \mathbb{Q}$ and $q = e^{\pi i \hbar}$, there is an abelian subcategory $\mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g})) \subset \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, and an exact, faithful functor*

$$\Gamma : \mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g})) \longrightarrow \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$$

which exponentiates the roots of Drinfeld polynomials, and commutes with the forgetful functor to vector spaces.

The category $\mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g}))$ of *non–congruent* representations is defined as follows. Let $\mathbf{I}$ be the set of nodes of the Dynkin diagram of $\mathfrak{g}$, and let $\{\xi_{i,r}, x_{i,r}^{\pm}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ be the loop generators of $Y_\hbar(\mathfrak{g})$ (see [16, 32] or §2). Consider the generating series

$$\xi_i(u) = 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-u-1} \qquad \text{and} \qquad x_i^{\pm}(u) = \hbar \sum_{r \geq 0} x_{i,r}^{\pm} u^{-r-1} \qquad (1.1)$$

We prove in §3.6 that, on a representation $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, these formal power series are the expansions near $u = \infty$ of $\mathrm{End}(V)$–valued rational functions. By definition, $\mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g}))$ consists of those representations such that, for each $i \in \mathbf{I}$, the poles of $x_i^+(u)$ (resp. $x_i^-(u)$) do not differ by non–zero integers.

1.7.   Our second result asserts the existence of a right inverse to Γ for each determination of the logarithm. More precisely, fix a subset $\Pi \subset \mathbb{C}$ such that

$$(\Pi - \Pi) \cap \mathbb{Z} = \{0\} \qquad \text{and} \qquad \Pi \pm \frac{\hbar}{2} \subset \Pi \qquad (1.2)$$

The map $u \mapsto e^{2\pi \iota u}$ is then a bijection between Π and a subset $\Omega \subset \mathbb{C}^\times$ stable under multiplication by q. To such a choice, we attach subcategories

$$\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g})) \subset \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g})) \qquad \text{and} \qquad \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g})) \subset \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$$

consisting of representations such that the roots of their Drinfeld polynomials lie in Π and Ω respectively, see §3.5. We show in §3 that $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$ are Serre subcategories of $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ that is, are closed under taking extensions, quotients and subobjects. We also show that V is in $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ if and only if the poles of $\xi_i(u)^{\pm 1}$ and $x_i^{\pm}(u)$ lie in Π. In particular, representations in $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ are not congruent.

Theorem. *The following holds for any choice of $\Pi \subset \mathbb{C}$ satisfying (1.2).*

(i) *The functor Γ restricts to a functor*

$$\Gamma_\Pi : \mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g})) \longrightarrow \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$$

(ii) *There is a functor*

$$\daleth_\Pi : \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g})) \longrightarrow \mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$$

which is a left and right inverse to Γ_Π, and commutes with the forgetful functor to vector spaces.

In particular, if Π is a fundamental domain for the translations $u \to u+1$, Γ_Π is an equivalence between $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$.

1.8.   The main novel idea underlying our construction is that each of the commuting fields $\xi_i(u)$ of $Y_\hbar(\mathfrak{g})$ may be used to define an *additive difference equation* on any representation $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$. The functor Γ is then governed by the monodromy of this equation, an averaging procedure which provides a canonical means of replacing the rational functions appearing in the representation theory of $Y_\hbar(\mathfrak{g})$ by their trigonometric counterparts appearing in that of $U_q(L\mathfrak{g})$ (for the reader's convenience, a self–contained account of Birkhoff's theory of difference equations is given in §4). It is worth stressing that these equations are *abelian*, so that their fundamental solutions are readily computable in terms of Γ–functions. [2] Γ is then defined using contour integral formulae using these solutions as kernels, and shown to define representations of $U_q(L\mathfrak{g})$ by contour manipulations reminiscent of those of Conformal Field Theory.

1.9.   The relevance of difference equations to Yangians and quantum loop algebras is widely known. For example, the R–matrix of $Y_\hbar(\mathfrak{g})$ (resp. $U_q(L\mathfrak{g})$) give rise to the additive (resp. multiplicative) qKZ equations satisfied by products of intertwiners [25, 47]. In a related vein, Baxter's Q–operator for $Y_\hbar(\mathfrak{g})$ (resp. $U_q(L\mathfrak{g})$) satisfies the TQ relations, which are difference equations with step a multiple of $\hbar$ (resp. a power of q) (see, *e.g.,* [1, 4, 23]). Moreover, the idea of averaging a rational or trigonometric solution of the quantum Yang–Baxter equations to obtain a trigonometric or elliptic solution has also been extensively used [2, 3, 20, 48].

It is worth pointing out that the difference equations used in this paper are not of qKZ or TQ type since they are abelian and therefore far simpler.

1.10.   The relevance of these abelian difference equations, and their use to define a functor between finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ is the main new ingredient of the present paper. This is a potentially far reaching principle, which we have so far applied to two further problems

(1) The construction of a tensor structure on the functor Γ, which yields a Kazhdan–Lusztig equivalence of meromorphic braided tensor categories between $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, when both are endowed with the deformed Drinfeld coproduct and the commutative part R_0 of the R–matrix. This tensor structure arises from the monodromy of the abelian qKZ equations defined by R_0 of $Y_\hbar(\mathfrak{g})$ [28].

(2) The constuction of a faithful functor Ξ between finite–dimensional representations of $U_q(L\mathfrak{g})$ and of Felder's elliptic quantum group [26]. The functor Ξ is governed by the monodromy of the q–difference equations defined by the commuting fields of $U_q(L\mathfrak{g})$. This is a very promising avenue, since elliptic quantum groups and their representations are still not very well understood outside of type A [22].

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[2]These Γ–functions are closely related to those appearing in [34, 45]

1.11. Let us outline the construction of the functor Γ, assuming for notational convenience that $\mathfrak{g}$ is semisimple. Let $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, and for any $i \in \mathbf{I}$, consider the additive difference equation

$$\phi_i(u + 1) = \xi_i(u)\phi_i(u) \tag{1.3}$$

determined by the field $\xi_i(u) = 1 + \hbar \xi_{i,0} u^{-1} + \cdots$ defined in (1.1), where the unknown ϕ_i is a meromorphic function of $u \in \mathbb{C}$ with values in $\mathrm{End}(V)$. As explained in §4, this equation admits two fundamental solutions $\phi_i^\pm(u)$, which are uniquely determined by the requirements that

$\bullet$ $\phi_i^\pm(u)$ is holomorphic and invertible for $\pm \mathrm{Re}(u) \gg 0$

$\bullet$ $\phi_i^\pm(u)$ is asymptotic to $(1 + O(u^{-1}))(\pm u)^{\hbar \xi_{i,0}}$ as $u \to \infty$ with $\pm \mathrm{Re}(u) \gg 0$

These solutions are regularisations of the formal infinite products

$$\xi_i(u)^{-1}\xi_i(u + 1)^{-1}\cdots \qquad \text{and} \qquad \xi_i(u - 1)\xi_i(u - 2)\cdots$$

which symbolically solve (1.3).

Following Birkhoff [6], one considers the connection or monodromy matrix of the equation, that is the $GL(V)$–valued function defined by

$$S_i(u) = \phi_i^+(u)^{-1} \cdot \phi_i^-(u)$$

By construction, $S_i(u)$ is a 1–periodic function of u. The asymptotics of $\phi_i^\pm$ further imply that S_i is a rational function of the variable $z = e^{2\pi\iota u}$, regular at $z = \infty, 0$, and such that $S_i(\infty) = e^{\pi\iota\hbar\xi_{i,0}} = S_i(0)^{-1}$.

1.12. Assuming that V is non–congruent, the action of the loop generators $\{\Psi_{i,\pm k}^\pm, \mathcal{X}_{i,l}^\pm\}_{i\in\mathbf{I}, k\in\mathbb{N}, l\in\mathbb{Z}}$ of $U_q(L\mathfrak{g})$ on $\Gamma(V) = V$ is now defined as follows.

(i) For each $i \in \mathbf{I}$, the generating series

$$\Psi_i(z)^\pm = \sum_{k \geq 0} \Psi_{i,\pm k}^\pm z^{\mp k} \tag{1.4}$$

acts as the Taylor series of the connection matrix $S_i(z)$ at $z = \infty, 0$.

(ii) For each $i \in \mathbf{I}$ and $k \in \mathbb{Z}$, $\mathcal{X}_{i,k}^\pm$ acts as the operator

$$c_i \oint_{\mathcal{C}_i^\pm} e^{2\pi\iota k u} g_i^\pm(u) x_i^\pm(u)\, du$$

where
- The constant c_i is given by $\sqrt{d_i}\,\Gamma(d_i\hbar)$, with d_i the symmetrising integers of the Cartan matrix of $\mathfrak{g}$.
- The functions $g_i^\pm(u)$ are regularisations of the infinite products

$$\cdots \xi_i(u + 2)\xi_i(u + 1) \qquad \text{and} \qquad \xi_i(u - 1)\xi_i(u - 2)\cdots$$

respectively, and are given in terms of the fundamental solutions $\phi_i^\pm(u)$ of (1.3) by

$$g_i^+(u) = \phi_i^+(u + 1)^{-1} \qquad \text{and} \qquad g_i^-(u) = \phi_i^-(u) \tag{1.5}$$

- The contour $\mathcal{C}_i^{\pm}$ contains all the poles of $x_i^{\pm}(u)$, and none of their $\mathbb{Z}^{\times}$–translates. [3]

1.13. The construction of the right inverse functor $\daleth_{\Pi} : \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g})) \to \mathcal{O}_{\mathrm{int}}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ is obtained by solving the inverse monodromy problem corresponding to the difference equation (1.3).

Specifically, let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$, and consider the $\mathrm{End}(\mathcal{V})$–valued generating series $\Psi_i(z)^{\pm}$ given by (1.4),

$$\mathcal{X}_i^{\pm}(z)^+ = \sum_{k\geq 0} \mathcal{X}_{i,k}^{\pm} z^{-k} \qquad \text{and} \qquad \mathcal{X}_i^{\pm}(z)^- = -\sum_{k\geq 1} \mathcal{X}_{i,-k}^{\pm} z^{k}$$

It is well–known that these are the expansions at $z = \infty, 0$ of rational functions $\Psi_i(z)$ and $\mathcal{X}_i^{\pm}(z)$ [5, 32] (see also Prop. 3.6). We show in §3 that $\mathcal{V}$ is in $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$ if, and only if the poles of $\Psi_i(z)^{\pm 1}$ and $\mathcal{X}_i^{\pm}(z)$ are contained in Ω.

The operators giving the action of $Y_{\hbar}(\mathfrak{g})$ on $\daleth_{\Pi}(\mathcal{V}) = \mathcal{V}$ are then defined as follows.

(i) As explained in §4, there is a unique rational function $\mathbb{C} \to GL(V)$, which we take to be $\xi_i(u)$, such that (a) $\xi_i(u) = 1 + \hbar d_i \alpha_i^{\vee} u^{-1} + \cdots$ (b) the connection matrix $S_i(z)$ of the difference equation (1.3) defined by $\xi_i(u)$ is $S_i(z) = \Psi_i(z)$ and (c) the poles of $\xi_i(u)^{\pm 1}$ lie in Π.

(ii) Let $\phi_i^{\pm}(u)$ be the canonical fundamental solutions of (1.3), and define the functions $g_i^{\pm}(u)$ by (1.5). Then, for any $i \in \mathbf{I}$ and $r \in \mathbb{N}$, the generators $x_{i,r}^{\pm}$ act on $\mathcal{V}$ as the operators

$$\frac{1}{c_i \hbar} \int_{\mathcal{C}_i^{\pm}} v^{-r} g_i^{\pm}(v)^{-1} \mathcal{X}_i^{\pm}\left(e^{2\pi\iota v}\right) dv$$

where the constants c_i are as in §1.12, and the contours $\mathcal{C}_i^{\pm}$ enclose all the poles of $\mathcal{X}_i^{\pm}\left(e^{2\pi\iota v}\right)$ contained in Π, none of their $\mathbb{Z}^{\times}$–translates, and none of the $\mathbb{Z}^{\times}$–translates of the poles of $\xi_i(u)^{-1}$.

1.14. The main construction of this paper can also be applied to the q–difference equations defined by the commuting fields of the quantum loop algebra, and yields a faithful functor between finite–dimensional representations of $U_q(L\mathfrak{g})$ and those of Felder's elliptic quantum group corresponding to $\mathfrak{g}$ [26]. We also plan to apply similar ideas to relate the affine Yangian $Y_{\hbar}(\widehat{\mathfrak{gl}_1})$ appearing in the recent work of Maulik–Okounkov [42] to the quantum toroïdal algebra $U_q(\widehat{\mathfrak{gl}_1})$ [21, 43, 46].

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[3]Note that the existence of the contours $\mathcal{C}_i^{\pm}$ relies on the representation V being non–congruent in the sense explained in §1.6.

1.15. The picture described in this paper is consistent with Cherednik's philosophy of obtaining equivalences of appropriate categories of finite–dimensional representations of the affine Hecke algebra $\mathbb{H}$ and degenerate affine Hecke algebra $\mathfrak{H}$ of a finite Coxeter group through the monodromy of a flat connection with values in $\mathfrak{H}$ [13, §1.1–1.2]. We note that, whereas Cherednik's affine KZ connection is differential and not abelian, our equations are difference and abelian. It seems an interesting problem to relate these two approaches.

1.16. **Outline of the paper.** In Section 2, we review the definitions of the Yangian $Y_\hbar(\mathfrak{g})$ and quantum loop algebra $U_q(L\mathfrak{g})$ associated to a symmetrisable Kac–Moody algebra $\mathfrak{g}$.

In Section 3, we consider the categories $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ of representations whose restriction to $\mathfrak{g}$ and $U_q\mathfrak{g}$ respectively is integrable and in category $\mathcal{O}$. We review the classification of their simple objects, and prove that the generating series of the loop generators of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ act on $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g})), \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ as rational functions. We then define the subcategories $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g})) \subset \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g})) \subset \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ in terms of the roots of Drinfeld polynomials. The main result of this section, Theorem 3.8, gives an alternative characterisation of these categories in terms of the poles of the fields $\xi_i(u), x_i^\pm(u)$ of $Y_\hbar(\mathfrak{g})$ (resp. $\Psi_i(z), \mathcal{X}_i^\pm(z)$ of $U_q(L\mathfrak{g})$).

Section 4 contains a self–contained account of the theory of additive difference equations including canonical fundamental solutions, monodromy and a detailed discussion of the inverse problem for abelian equations.

Section 5 gives the definition of the functor $\Gamma : \mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g})) \to \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ and the proof that the operators defined in §1.12 give an action of $U_q(L\mathfrak{g})$ on any non–congruent representation of $Y_\hbar(\mathfrak{g})$. We also prove the compatibility of Γ with the shift automorphisms of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$.

Section 6 contains the contruction of the inverse functor $\daleth_\Pi : \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g})) \to \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$, and the proof that $\daleth_\Pi \circ \Gamma_\Pi$ and $\Gamma_\Pi \circ \daleth_\Pi$ are the identity functors on $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$ respectively.

In Section 7, we review the definition of the q–characters of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, and show that the functor Γ is compatible with these.

2. Yangians and quantum loop algebras

2.1.   Let $\mathbf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ be a symmetrisable generalized Cartan matrix [36]. Thus, $a_{ii} = 2$ for any $i \in \mathbf{I}$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ for any $i \neq j \in \mathbf{I}$, and there exists a diagonal matrix D with positive integer entries $\{d_i\}_{i\in\mathbf{I}}$ such that DA is symmetric. We assume that (d_i) are relatively prime.

Let $(\mathfrak{h}, \{\alpha_i\}_{i\in\mathbf{I}}, \{\alpha_i^\vee\}_{i\in\mathbf{I}})$ be the unique realization of $\mathbf{A}$. Thus, $\mathfrak{h}$ is a complex vector space of dimension $2|\mathbf{I}| - \mathrm{rank}(\mathbf{A})$, $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee\}_{i\in\mathbf{I}} \subset \mathfrak{h}$ are linearly independent sets and, for any $i, j \in \mathbf{I}$, $\alpha_j(\alpha_i^\vee) = a_{ij}$.

Let $\mathfrak{g}$ be the Kac–Moody algebra associated to $\mathbf{A}$.

2.2. **The Yangian $Y_\hbar(\mathfrak{g})$.** Let $\hbar \in \mathbb{C}$. The Yangian $Y_\hbar(\mathfrak{g})$ is the $\mathbb{C}$–algebra generated by elements $\{x_{i,r}^\pm, \xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ and $h \in \mathfrak{h}$, subject to the following relations

(Y0) For any $i \in \mathbf{I}$,
$$\xi_{i,0} = d_i\alpha_i^\vee$$

(Y1) For any $i, j \in \mathbf{I}$, $r, s \in \mathbb{N}$ and $h, h' \in \mathfrak{h}$
$$[\xi_{i,r}, \xi_{j,s}] = 0 \qquad\qquad [\xi_{i,r}, h] = 0 \qquad\qquad [h, h'] = 0$$

(Y2) For $h \in \mathfrak{h}$, $j \in \mathbf{I}$ and $s \in \mathbb{N}$
$$[h, x_{j,s}^\pm] = \pm\alpha_j(h)x_{j,s}^\pm$$

(Y3) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$
$$[\xi_{i,r+1}, x_{j,s}^\pm] - [\xi_{i,r}, x_{j,s+1}^\pm] = \pm\hbar\frac{d_i a_{ij}}{2}(\xi_{i,r}x_{j,s}^\pm + x_{j,s}^\pm\xi_{i,r})$$

(Y4) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$
$$[x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm\hbar\frac{d_i a_{ij}}{2}(x_{i,r}^\pm x_{j,s}^\pm + x_{j,s}^\pm x_{i,r}^\pm)$$

(Y5) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$
$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij}\xi_{i,r+s}$$

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \cdots, r_m \in \mathbb{N}$ and $s \in \mathbb{N}$
$$\sum_{\pi\in\mathfrak{S}_m}\left[x_{i,r_{\pi(1)}}^\pm, \left[x_{i,r_{\pi(2)}}^\pm, \left[\cdots, \left[x_{i,r_{\pi(m)}}^\pm, x_{j,s}^\pm\right]\cdots\right]\right]\right] = 0$$

2.3.   Assume now that $\hbar \neq 0$, and define $\xi_i(u), x_i^\pm(u) \in Y_\hbar(\mathfrak{g})[[u^{-1}]]$ by
$$\xi_i(u) = 1 + \hbar\sum_{r\geq 0}\xi_{i,r}u^{-r-1} \qquad \text{and} \qquad x_i^\pm(u) = \hbar\sum_{r\geq 0}x_{i,r}^\pm u^{-r-1}$$

Proposition. *The relations (Y1),(Y2)–(Y3),(Y4),(Y5),(Y6) are equivalent to the following identities in $Y_\hbar(\mathfrak{g})[u, v; u^{-1}, v^{-1}]]$*

($\mathcal{Y}$1) *For any $i, j \in \mathbf{I}$ and $h, h' \in \mathfrak{h}$,*
$$[\xi_i(u), \xi_j(v)] = 0 \qquad\qquad [\xi_i(u), h] = 0 \qquad\qquad [h, h'] = 0$$

($\mathcal{Y}2$) *For any i, and $h \in \mathfrak{h}$,*

$$[h, x_i^{\pm}(u)] = \pm\alpha_i(h)x_i^{\pm}(u)$$

($\mathcal{Y}3$) *For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij}/2$*

$$(u - v \mp a)\xi_i(u)x_j^{\pm}(v) = (u - v \pm a)x_j^{\pm}(v)\xi_i(u) \mp 2ax_j^{\pm}(u \mp a)\xi_i(u)$$

($\mathcal{Y}4$) *For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij}/2$*

$$(u - v \mp a)x_i^{\pm}(u)x_j^{\pm}(v)$$
$$= (u - v \pm a)x_j^{\pm}(v)x_i^{\pm}(u) + \hbar\left([x_{i,0}^{\pm}, x_j^{\pm}(v)] - [x_i^{\pm}(u), x_{j,0}^{\pm}]\right)$$

($\mathcal{Y}5$) *For any $i, j \in \mathbf{I}$*

$$(u - v)[x_i^+(u), x_j^-(v)] = -\delta_{ij}\hbar\left(\xi_i(u) - \xi_i(v)\right)$$

($\mathcal{Y}6$) *For any $i \neq j \in \mathbf{I}$, $m = 1 - a_{ij}$, $r_1, \cdots, r_m \in \mathbb{N}$, and $s \in \mathbb{N}$*

$$\sum_{\pi \in \mathfrak{S}_m}\left[x_i^{\pm}(u_{\pi_1}), \left[x_i^{\pm}(u_{\pi(2)}), \left[\cdots, \left[x_i^{\pm}(u_{\pi(m)}), x_j^{\pm}(v)\right]\cdots\right]\right]\right] = 0$$

Remark. When $i = j$, ($\mathcal{Y}4$) may be rewritten as follows. Taking $u = v$ yields

$$[x_{i,0}^{\pm}, x_i^{\pm}(u)] = \mp d_i x_i^{\pm}(u)^2 \tag{2.1}$$

and therefore the relation ($\mathcal{Y}4'$)

$$(u - v \mp \hbar d_i)x_i^{\pm}(u)x_i^{\pm}(v)$$
$$= (u - v \pm \hbar d_i)x_i^{\pm}(v)x_i^{\pm}(u) \mp \hbar d_i\left(x_i^{\pm}(u)^2 + x_i^{\pm}(v)^2\right)$$

Conversely, equating coefficients of v^0 in this identity yields (2.1), and therefore ($\mathcal{Y}4$).

2.4. **Proof of Proposition 2.3.** The equivalences (Y1)≡($\mathcal{Y}1$) and (Y6)≡($\mathcal{Y}6$) are clear.

To see that (Y4) implies ($\mathcal{Y}4$), multiply (Y4) by $\hbar u^{-r-1}\hbar v^{-s-1}$. Summing over $r, s \geq 0$, and using $\hbar \sum_{r\geq 0} x_{i,r+1}^{\pm} u^{-r-1} = ux_i^{\pm}(u) - \hbar x_{i,0}^{\pm}$ yields

$$(u - v)[x_i^{\pm}(u), x_j^{\pm}(v)] - \hbar\left([x_{i,0}^{\pm}, x_j^{\pm}(v)] - [x_i^{\pm}(u), x_{j,0}^{\pm}(v)]\right)$$
$$= \pm a\left(x_i^{\pm}(u)x_j^{\pm}(v) + x_j^{\pm}(v)x_i^{\pm}(u)\right)$$

as claimed. Conversely, taking coefficients of $u^{-r-1}v^{-s-1}$ in ($\mathcal{Y}4$) for $r, s \geq 0$ yields (Y4).[4]

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[4]equating coefficients of $u^0 v^{-s-1}$, and of $u^{-r-1}v^0$ yields the tautological identities $[x_{i,0}^{\pm}, x_{j,s}^{\pm}] = [x_{i,0}^{\pm}, x_{j,s}^{\pm}]$ and $[x_{i,r}^{\pm}, x_{j,0}^{\pm}] = [x_{i,r}^{\pm}, x_{j,0}^{\pm}]$ respectively.

(Y2) is clearly equivalent to ($\mathcal{Y}$2). To see that (Y2)–(Y3) imply ($\mathcal{Y}$3), set $\xi_{i,-1} = \hbar^{-1}$ so that

$$\xi_i(u) = \hbar \sum_{r \geq -1} \xi_{i,r} u^{-r-1}$$

and (Y2) for $h = \xi_{i,0}$ coincides with (Y3) for $r = -1$. Multiplying both sides of (Y3) by $\hbar u^{-r-1} \hbar v^{-s-1}$, and summing over $r \geq -1, s \geq 0$ yields

$$(u - v \mp a)\xi_i(u)x_j^{\pm}(v) - (u - v \pm a)x_j^{\pm}(v)\xi_i(u) = -\hbar[\xi_i(u), x_{j,0}^{\pm}]$$

Since the right–hand side is independent of v, so is the left–hand side. The latter is therefore equal to its value when $v = u \mp a$, that is $\mp 2a x_j^{\pm}(u \mp a)\xi_i(u)$ and ($\mathcal{Y}$3) follows. Conversely, taking coefficients of $u^{-r-1}v^{-s-1}$, $r, s \geq 0$ in ($\mathcal{Y}$3) yields (Y3)[5]

To see that (Y5) implies ($\mathcal{Y}$5), multiply (Y5) by $\hbar u^{-r-1} \hbar v^{-s-1}$. Summing over $r, s \geq 0$, and using

$$(u - v) \sum_{r+s=t} u^{-r-1}v^{-s-1} = v^{-t-1} - u^{-t-1}$$

then yields ($\mathcal{Y}$5). Conversely, the above identity implies that $\xi_i(u) - \xi_i(v) = \hbar(v - u) \sum_{r,s \geq 0} u^{-r-1}v^{-s-1}\xi_{i,r+s}$. Equating this to $\hbar^{-1}(v - u)[x_i^+(u), x_i^-(v)]$ yields (Y5) since $\mathbb{C}[u, v; u^{-1}, v^{-1}]]$ has no zero divisors.

2.5.   The following result is due to Levendorskii [39]. We reproduce his proof below for completeness.

Lemma. *The relation (Y6) follows from (Y1)–(Y3) and the special case of (Y6) when $r_1 = \cdots = r_m = 0$.*

PROOF. The proof of this lemma is based on the construction of commuting elements $\{\widetilde{t}_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$. These are polynomials in $\{\xi_{i,r}\}$ such that the following commutation relations hold

$$[\widetilde{t}_{i,r}, x_{i,s}^{\pm}] = \pm 2d_i x_{i,r+s}^{\pm}$$

$$[\widetilde{t}_{i,r}, x_{j,s}^{\pm}] = \sum_{k=0}^{r+s} b_{ij}^{\pm}(r, s, k)x_{j,r+s-k}^{\pm}$$

where $b_{ij}^{\pm}(r, s, k) \in \mathbb{C}$ are scalars. We postpone their construction to 2.6, and proceed with the proof of the lemma.

Denote the left–hand side of (Y6) by $S_{ij}^{\pm}(r_1, \cdots, r_m; s)$. We prove that this element is zero by induction on the number of non–zero r_j. The case $S_{ij}^{\pm}(0, \cdots, 0; s) = 0$ holds by assumption. Assume now that

$$S_{ij}^{\pm}(r_1, \cdots, r_k, 0, \cdots, 0; s) = 0$$

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[5]equating coefficients of u^1, u^0 yields the identities $x_j^{\pm}(v) = x_j^{\pm}(v)$ and (Y2) for $h = \xi_{i,0}$ respectively, while equating coefficients of v^0 yields the identity $-\hbar[\xi_i(u), x_{j,0}^{\pm}] = \mp 2a x_j^{\pm}(u \mp a)\xi_i(u)$.

for some $0 \leq k < m$ and for every $r_1, \cdots, r_k, s \in \mathbb{N}$. Applying $\mathrm{ad}(\widetilde{t}_{i,r})$ to this identity yields

$$0 = \pm 2d_i(m-k)S_{ij}^{\pm}(r_1, \cdots, r_k, r, 0, \cdots, 0; s)$$

$$\pm 2d_i \sum_{p=1}^{k} S_{ij}^{\pm}(r_1, \cdots, r_{p-1}, r_p + r, r_{p+1}, \cdots, r_k, 0, \cdots, 0; s)$$

$$+ \sum_{p=0}^{r+s} b_{ij}^{\pm}(r, s, p)S_{ij}^{\pm}(r_1, \cdots, r_k, 0, \cdots, 0; r+s-p)$$

Since the terms in the last two sums are zero by induction, it follows that $S_{ij}^{\pm}(r_1, \cdots, r_k, r, 0, \cdots, 0; s) = 0$. $\qquad\square$

2.6. Define $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ by

$$\hbar \sum_{r \geq 0} t_{i,r} u^{-r-1} = \log\left(\xi_i(u)\right)$$

The following commutation relation was obtained in [39, Lemma 1.9] (see also [27, Remark 2.9])

$$[t_{i,r}, x_{j,s}^{\pm}] = \pm d_i a_{ij} \sum_{l=0}^{\lfloor r/2 \rfloor} \binom{r}{2l} \frac{(\hbar d_i a_{ij}/2)^{2l}}{2l+1} x_{j,r+s-2l}^{\pm}$$

Now it is clear that one can define $\widetilde{t}_{i,r}$ as

$$\widetilde{t}_{i,r} = t_{i,r} + \text{ linear combination of } t_{i,r-k} \ (1 \leq k \leq r)$$

such that $[\widetilde{t}_{i,r}, x_{i,s}^{\pm}] = \pm 2d_i x_{i,r+s}^{\pm}$. The commutation relation above, and the lower triangularity of the transformation matrix between $\{\widetilde{t}_{i,r}\}$ and $\{t_{i,r}\}$ implies that we also have

$$[\widetilde{t}_{i,r}, x_{j,s}^{\pm}] = \sum_{p=0}^{r+s} b_{ij}^{\pm}(r, s, p) x_{j,r+s-p}^{\pm}$$

for some scalars $b_{ij}^{\pm}(r, s, p)$.

2.7. It follows from Lemma 2.5 that, on an integrable representation, relation (Y6) can be deduced from the other relations. More precisely, let $\widetilde{Y}_{\hbar}(\mathfrak{g})$ be a unital associative algebra generated by $\{h, \xi_{i,r}, x_{i,r}^{\pm}\}$ subject to the relations (Y0)–(Y5) of Section 2.2. Let V be a representation of $\widetilde{Y}_{\hbar}(\mathfrak{g})$ which is integrable in the following sense

- V is $\mathfrak{h}$–diagonalizable with finite–dimensional weight spaces.
- The action of $\{x_{i,0}^{\pm}\}$ on V is locally nilpotent.

Proposition. *The action of $\widetilde{Y}_{\hbar}(\mathfrak{g})$ on V factors through one of $Y_{\hbar}(\mathfrak{g})$.*

PROOF. By Lemma 2.5, it suffices to prove that relation (Y6) with $r_1 = \cdots = r_m = 0$ holds in $\mathrm{End}(V)$. Let $\mathfrak{sl}_2^i \subset Y_\hbar(\mathfrak{g})$ be the Lie subalgebra generated by $\{\xi_{i,0}, x_{i,0}^\pm\}$. Let $V[\mu]$, $\mu \in \mathfrak{h}^*$ be a weight subspace of V, and set

$$\overline{V} = \bigoplus_{m \in \mathbb{Z}} V[\mu + m\alpha_i] \qquad \text{and} \qquad \overline{V}_j^\pm = \bigoplus_{m \in \mathbb{Z}} V[\mu + m\alpha_i \pm \alpha_j]$$

$\overline{V}, \overline{V}_j^\pm$ are $\mathfrak{sl}_2^i$–invariant subspaces of V which, by the integrability of V, are finite–dimensional. Since $[\xi_{i,0}, x_{j,s}^\pm] = \pm d_i a_{ij} x_{j,s}^\pm$ and $\mathrm{ad}(x_{i,0}^\mp)x_{j,s}^\pm = 0$, the restriction of $x_{j,s}^\pm$ to $\overline{V}$ is a lowest (resp. highest) weight vector in the finite–dimensional $\mathfrak{sl}_2^i$–module $\mathrm{Hom}(\overline{V}, \overline{V}_j^\pm)$, which implies that $\mathrm{ad}(x_{i,0}^\pm)^{1-a_{ij}} x_{j,s}^\pm$ acts by 0 on $\overline{V}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

2.8. Shift automorphism.

The group of translations of the complex plane acts on $Y_\hbar(\mathfrak{g})$ by

$$\tau_a(y_r) = \sum_{s=0}^r \binom{r}{s} a^{r-s} y_s \qquad \text{and} \qquad \tau_a(h) = h$$

where $a \in \mathbb{C}$, y is one of $\xi_i, x_i^\pm$ and $h \in \mathfrak{h}$. In terms of the generating series introduced in 2.3,

$$\tau_a(y(u)) = y(u - a)$$

Given a representation V of $Y_\hbar(\mathfrak{g})$ and $a \in \mathbb{C}$, we set $V(a) = \tau_a^*(V)$.

2.9. The quantum loop algebra $U_q(L\mathfrak{g})$.

Let $q \in \mathbb{C}^\times$ be of infinite order. For any $i \in \mathbf{I}$, set $q_i = q^{d_i}$. We use the standard notation for Gaussian integers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

The quantum loop algebra $U_q(L\mathfrak{g})$ is the $\mathbb{C}$–algebra generated by elements $\{\Psi_{i,\pm r}^\pm\}_{i \in \mathbf{I}, r \in \mathbb{N}}$, $\{\mathcal{X}_{i,k}^\pm\}_{i \in \mathbf{I}, k \in \mathbb{Z}}$ and $\{K_h\}_{h \in \mathfrak{h}}$, subject to the following relations

(QL0) For any $i \in \mathbf{I}$
$$\Psi_{i,0}^\pm = K_{\pm d_i \alpha_i^\vee}$$

(QL1) For any $i, j \in \mathbf{I}$, $r, s \in \mathbb{N}$ and $h, h' \in \mathfrak{h}$,
$$[\Psi_{i,\pm r}^\pm, \Psi_{j,\pm s}^\pm] = 0 \qquad [\Psi_{i,\pm r}^\pm, \Psi_{j,\mp s}^\mp] = 0 \qquad [\Psi_{i,\pm r}^\pm, K_h] = 0$$
$$K_h K_{h'} = K_{h+h'} \qquad K_0 = 1$$

(QL2) For any $i \in \mathbf{I}$, $k \in \mathbb{Z}$ and $h \in \mathfrak{h}$,
$$K_h \mathcal{X}_{i,k}^\pm K_h^{-1} = q^{\pm \alpha_i(h)} \mathcal{X}_{i,k}^\pm$$

(QL3) For any $i, j \in \mathbf{I}$, $\varepsilon \in \{\pm\}$ and $l \in \mathbb{Z}$

$$\Psi^{\varepsilon}_{i,k+1}\mathcal{X}^{\pm}_{j,l} - q^{\pm a_{ij}}_i \mathcal{X}^{\pm}_{j,l}\Psi^{\varepsilon}_{i,k+1} = q^{\pm a_{ij}}_i \Psi^{\varepsilon}_{i,k}\mathcal{X}^{\pm}_{j,l+1} - \mathcal{X}^{\pm}_{j,l+1}\Psi^{\varepsilon}_{i,k}$$

for any $k \in \mathbb{Z}_{\geq 0}$ if $\varepsilon = +$ and $k \in \mathbb{Z}_{<0}$ if $\varepsilon = -$

(QL4) For any $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$

$$\mathcal{X}^{\pm}_{i,k+1}\mathcal{X}^{\pm}_{j,l} - q^{\pm a_{ij}}_i \mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k+1} = q^{\pm a_{ij}}_i \mathcal{X}^{\pm}_{i,k}\mathcal{X}^{\pm}_{j,l+1} - \mathcal{X}^{\pm}_{j,l+1}\mathcal{X}^{\pm}_{i,k}$$

(QL5) For any $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$

$$[\mathcal{X}^+_{i,k}, \mathcal{X}^-_{j,l}] = \delta_{ij}\frac{\Psi^+_{i,k+l} - \Psi^-_{i,k+l}}{q_i - q_i^{-1}}$$

where we set $\Psi^{\pm}_{i,\mp k} = 0$ for any $k \geq 1$.

(QL6) For any $i \neq j \in \mathbf{I}$, $m = 1 - a_{ij}$, $k_1, \ldots, k_m \in \mathbb{Z}$ and $l \in \mathbb{Z}$

$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^{m}(-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \mathcal{X}^{\pm}_{i,k_{\pi(1)}} \cdots \mathcal{X}^{\pm}_{i,k_{\pi(s)}} \mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k_{\pi(s+1)}} \cdots \mathcal{X}^{\pm}_{i,k_{\pi(m)}} = 0$$

2.10. Define $\Psi_i(z)^+, \mathcal{X}^{\pm}_i(z)^+ \in U_q(L\mathfrak{g})[[z^{-1}]]$ and $\Psi_i(z)^-, \mathcal{X}^{\pm}_i(z)^- \in U_q(L\mathfrak{g})[[z]]$ by

$$\Psi_i(z)^+ = \sum_{r \geq 0} \Psi^+_{i,r}z^{-r} \qquad\qquad \Psi_i(z)^- = \sum_{r \leq 0} \Psi^-_{i,r}z^{-r}$$

$$\mathcal{X}^{\pm}_i(z)^+ = \sum_{r \geq 0} \mathcal{X}^{\pm}_{i,r}z^{-r} \qquad\qquad \mathcal{X}^{\pm}_i(z)^- = -\sum_{r < 0} \mathcal{X}^{\pm}_{i,r}z^{-r}$$

Proposition. *The relations (QL1),(QL2)–(QL3),(QL4),(QL5),(QL6) imply the following relations in $U_q(L\mathfrak{g})[z, w; z^{-1}, w^{-1}]]$*

($\mathcal{QL}$1) *For any $i, j \in \mathbf{I}$, and $h, h' \in \mathfrak{h}$,*

$$[\Psi_i(z)^+, \Psi_j(w)^+] = 0 \qquad [\Psi_i(z)^+, K_h] = 0$$

$$K_h K_{h'} = K_{h+h'} \qquad K_0 = 1$$

($\mathcal{QL}$2) *For any $i \in \mathbf{I}$, and $h \in \mathfrak{h}$,*

$$K_h \mathcal{X}^{\pm}_i(z)^+ K_h^{-1} = q^{\pm \alpha_i(h)} \mathcal{X}^{\pm}_i(z)^+$$

($\mathcal{QL}$3) *For any $i, j \in \mathbf{I}$*

$$(z - q^{\pm a_{ij}}_i w)\Psi_i(z)^+\mathcal{X}^{\pm}_j(w)^+$$
$$= (q^{\pm a_{ij}}_i z - w)\mathcal{X}^{\pm}_j(w)^+\Psi_i(z)^+ - (q^{\pm a_{ij}}_i - q^{\mp a_{ij}}_i)q^{\pm a_{ij}}_i w \mathcal{X}^{\pm}_j(q^{\mp a_{ij}}_i z)^+\Psi_i(z)^+$$

($\mathcal{QL}$4) *For any $i, j \in \mathbf{I}$*

$$(z - q^{\pm a_{ij}}_i w)\mathcal{X}^{\pm}_i(z)^+\mathcal{X}^{\pm}_j(w)^+ - (q^{\pm a_{ij}}_i z - w)\mathcal{X}^{\pm}_j(w)^+\mathcal{X}^{\pm}_i(z)^+$$
$$= z\left(\mathcal{X}^{\pm}_{i,0}\mathcal{X}^{\pm}_j(w)^+ - q^{\pm a_{ij}}_i \mathcal{X}^{\pm}_j(w)^+\mathcal{X}^{\pm}_{i,0}\right) + w\left(\mathcal{X}^{\pm}_{j,0}\mathcal{X}^{\pm}_i(z)^+ - q^{\pm a_{ij}}_i \mathcal{X}^{\pm}_i(z)^+\mathcal{X}^{\pm}_{j,0}\right)$$

(QL5) *For any* $i, j \in \mathbf{I}$

$$(z - w)[\mathcal{X}_i^+(z)^+, \mathcal{X}_j^-(w)^+] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(z\Psi_i(w)^+ - w\Psi_i(z)^+ - (z - w)\Psi_{i,0}^- \right)$$

(QL6) *For any* $i \neq j \in \mathbf{I}$, *and* $m = 1 - a_{ij}$

$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \mathcal{X}_i^\pm(z_{\pi(1)})^+ \cdots \mathcal{X}_i^\pm(z_{\pi(s)})^+ \mathcal{X}_j^\pm(w)^+$$
$$\cdot \mathcal{X}_i^\pm(z_{\pi(s+1)})^+ \cdots \mathcal{X}_i^\pm(z_{\pi(m)})^+ = 0$$

Remark. When $i = j$, (QL4) may be rewritten as follows. Taking $z = w$ yields

$$\left(\mathcal{X}_{i,0}^\pm \mathcal{X}_i^\pm(z) - q_i^{\pm 2} \mathcal{X}_i^\pm(z)\mathcal{X}_{i,0}^\pm \right) = (1 - q_i^{\pm 2})\mathcal{X}_i^\pm(z)^2 \qquad (2.2)$$

and therefore the relation (QL4′)

$$(z - q_i^{\pm 2} w)\mathcal{X}_i^\pm(z)\mathcal{X}_i^\pm(w) - (q_i^{\pm 2} z - w)\mathcal{X}_i^\pm(w)\mathcal{X}_i^\pm(z)$$
$$= z(1 - q_i^{\pm 2})\mathcal{X}_i^\pm(w)^2 + w(1 - q_i^{\pm 2})\mathcal{X}_i^\pm(z)^2$$

Conversely, equating coefficients of z in this identity yields (2.2) and therefore (QL4).

2.11. **Proof of Proposition 2.10.** The implications (QLn)$\Rightarrow$(QLn), for $n = 1, 2, 6$ are clear.

Multiplying (QL4) by $z^{-k}w^{-l}$, summing over $k, l \geq 0$ and using the fact that $\sum_{k \geq 0} \mathcal{X}_{i,k+1}^\pm z^{-k} = z(\mathcal{X}_i^\pm(z) - \mathcal{X}_{i,0}^\pm)$ yields (QL4).

Multiplying (QL3) by $z^{-k}w^{-l}$, summing over $k, l \geq 0$ yields similarly

$$(z - q_i^{\pm a_{ij}} w)\Psi_i(z)^+ \mathcal{X}_j^\pm(w)^+ - (q_i^{\pm a_{ij}} z - w)\mathcal{X}_j^\pm(w)^+ \Psi_i(z)^+$$
$$= z(\Psi_0 \mathcal{X}_j^\pm(w)^+ - q_i^{\pm a_{ij}} \mathcal{X}_j^\pm(w)^+ \Psi_0) - w(q_i^{\pm a_{ij}} \Psi_i(z)^+ \mathcal{X}_{j,0}^\pm - \mathcal{X}_{j,0}^\pm \Psi_i(z)^+)$$

Since the first summand on the right–hand side is equal to 0 by (QL3), the ratio of the left–hand side by w is independent of w. It is therefore equal to its value for $w = q_i^{\mp a_{ij}} z$, that is $-q_i^{\pm a_{ij}}(q_i^{\pm a_{ij}} z - w)\mathcal{X}_j^\pm(q_i^{\mp a_{ij}} z)\Psi_i(z)$ and (QL4) holds.

Multiplying (QL5) by $(z - w)z^{-k}w^{-l}$, summing over $k, l \geq 0$, and using $(z - w)\sum_{k+l=m} z^{-k}w^{-l} = zw^{-m} - wz^{-m}$ yields (QL5).

Remark. The relations (QL1)–(QL6) of Proposition 2.10 for $\{\Psi_i(z)^-, \mathcal{X}_i^\pm(z)^-\}$ continue to hold in $U_q(L\mathfrak{g})[[z, w]]$. Similarly these relations remain true between $+$ and $-$ fields, in the appropriate algebra. For example, (QL3) for $\Psi_i(z)^+$ and $\mathcal{X}_i^\pm(w)^-$ is true in $U_q(L\mathfrak{g})[z, w^{-1}; z^{-1}, w]]$.

2.12.   The following result is the analogue of Lemma 2.5 for $U_q(L\mathfrak{g})$.

Lemma. *The relation (QL6) follows from (QL1)–(QL3) and the special case of (QL6) when* $k_1 = \cdots = k_m = 0$.

PROOF. We proceed as in Lemma 2.5. Define elements $\{H_{i,k}\}_{i\in\mathbf{I},k\in\mathbb{Z}^\times} \in U_q(L\mathfrak{g})$ by

$$\Psi_i(z)^\pm = \Psi_{i,0}^\pm \exp\left(\pm(q_i - q_i^{-1})\sum_{n\geq 1} H_{i,\pm n}z^{\mp n}\right)$$

Then we have the following relation

$$[H_{i,k}, \mathcal{X}_{j,l}^\pm] = \pm\frac{[ka_{ij}]_{q_i}}{k}\mathcal{X}_{j,k+l}^\pm$$

The proof then proceeds by an induction argument, as in that of Lemma 2.5. $\square$

2.13.   Let $\widetilde{U}_q(L\mathfrak{g})$ be the unital associative algebra generated by elements $\{K_h, \Psi_{i,\pm r}^\pm, \mathcal{X}_{i,l}^\pm\}$ subject to the relations (QL0)–(QL5) of Section 2.9. Let $\mathcal{V}$ be a representation of $\widetilde{U}_q(L\mathfrak{g})$ which is integrable in the following sense

- $\mathcal{V}$ is $\mathfrak{h}$–diagonalizable with finite–dimensional weight spaces.
- For each $i \in \mathbf{I}$, the action of $\mathcal{X}_{i,0}^\pm$ on $\mathcal{V}$ is locally nilpotent.

The proof of the following result is similar to that of Proposition 2.7 and therefore omitted.

Proposition. *The action of $\widetilde{U}_q(L\mathfrak{g})$ on $\mathcal{V}$ descends to that of $U_q(L\mathfrak{g})$.*

2.14. **Shift automorphism.** The group $\mathbb{C}^*$ of dilations of the complex plane acts on $U_q(L\mathfrak{g})$ by

$$\tau_\alpha(Y_k) = \alpha^k Y_k \qquad \text{and} \qquad \tau_\alpha(K_h) = K_h$$

where $\alpha \in \mathbb{C}^*$, Y is one of $\Psi_i^\pm, \mathcal{X}_i^\pm$ and $h \in \mathfrak{h}$. In terms of the generating series of 2.10, we have

$$\tau_\alpha(Y(z))^\pm = Y(\alpha^{-1}z)^\pm$$

Given a representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ and $\alpha \in \mathbb{C}^\times$, we denote $\tau_\alpha^*(\mathcal{V})$ by $\mathcal{V}(\alpha)$.

3. INTEGRABLE REPRESENTATIONS OF $Y_\hbar(\mathfrak{g})$ AND $U_q(L\mathfrak{g})$

3.1. **Integrable representations** [32]**.** By definition, a representation of $U_q(L\mathfrak{g})$ (resp. $Y_\hbar(\mathfrak{g})$) is in $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ (resp. $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$) if, and only if its restriction to $U_q(\mathfrak{g})$ (resp. $\mathfrak{g}$) is an integrable representation in the corresponding category $\mathcal{O}$ (see [36, 41]). Specifically,

(i) A representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ is in $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$ if
   (a) $\mathcal{V} = \bigoplus_{\mu\in\mathfrak{h}^*} \mathcal{V}_\mu$ and $\dim(\mathcal{V}_\mu) < \infty$, where

   $$\mathcal{V}_\mu = \{v \in \mathcal{V}\,|\, K_h v = q^{\mu(h)}v \text{ for every } h \in \mathfrak{h}\}$$

   (b) There exist $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ such that $\mathcal{V}_\mu \neq 0$ implies that $\mu \leq \lambda_i$ for some $i = 1, \ldots, r$. Recall that we say $\mu \leq \lambda$ if $\lambda - \mu \in \sum_{i\in\mathbf{I}} \mathbb{N}\alpha_i$.

(c) For each $\mu \in \mathfrak{h}^*$ such that $\mathcal{V}_\mu \neq 0$ and $i \in \mathbf{I}$, there exists $N > 0$ such that $\mathcal{V}_{\mu - n\alpha_i} = 0$ for every $n \geq N$.

(ii) A representation V of $Y_\hbar(\mathfrak{g})$ is in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ if

(a) $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ and $\dim(V_\mu) < \infty$, where

$$V_\mu = \{v \in V \,|\, hv = \mu(h)v \text{ for every } h \in \mathfrak{h}\}$$

(b) There exist $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ such that $V_\mu \neq 0$ implies that $\mu \leq \lambda_i$ for some $i = 1, \ldots, r$.

(c) For each $\mu \in \mathfrak{h}^*$ such that $V_\mu \neq 0$ and $i \in \mathbf{I}$, there exists $N > 0$ such that $V_{\mu - n\alpha_i} = 0$ for every $n \geq N$.

Remark. When $\mathfrak{g}$ is a semisimple Lie algebra, the categories $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ are the categories of finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$. Note that in this case, the definition of an integrable representations in category $\mathcal{O}$ for $U_q(L\mathfrak{g})$ given above differs from the usual one for $U_q\hat{\mathfrak{g}}$ given in [36, 41].

3.2. **Drinfeld polynomials: Yangians.** We recall below the classification of irreducible representations in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$. For $\mathfrak{g} = \mathfrak{sl}_2$, this result is proved in [9]. For the general case, a proof can be obtained by adapting the arguments of [12], see [32] and [11, Chaper 12].

Let $\mathbf{c} = \{c_{i,r}\}_{r \in \mathbb{N}, i \in \mathbf{I}} \subset \mathbb{C}$ be a collection of complex numbers, and let $\lambda \in \mathfrak{h}^*$ such that $c_{i,0} = d_i \lambda(\alpha_i^\vee)$. A representation V of $Y_\hbar(\mathfrak{g})$ is said to be a highest weight representation of highest weight $(\lambda, \mathbf{c})$ if there exists $\mathbf{v} \in V$ such that

(i) $V = Y_\hbar(\mathfrak{g})\mathbf{v}$.

(ii) $x_i^+(u)\mathbf{v} = 0$ for any $i \in \mathbf{I}$.

(iii) $\xi_{i,r}\mathbf{v} = c_{i,r}\mathbf{v}$ and $h\,\mathbf{v} = \lambda(h)\mathbf{v}$, for any $i \in \mathbf{I}, r \in \mathbb{N}$ and $h \in \mathfrak{h}$.

To the pair $(\lambda, \mathbf{c})$, we associate the Verma module $M(\lambda, \mathbf{c})$ in an obvious way and denote by $L(\lambda, \mathbf{c})$ its unique irreducible quotient.

Theorem.

(i) *Every irreducible representation in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ is a highest weight representation for a unique highest weight $(\lambda, \mathbf{c})$.*

(ii) *The irreducible representation $L(\lambda, \mathbf{c})$ is in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ if, and only if there exist (unique) monic polynomials $\{P_i(u) \in \mathbb{C}[u]\}_{i \in \mathbf{I}}$ such that*

$$1 + \hbar \sum_{r \geq 0} c_{i,r} u^{-r-1} = \frac{P_i(u + d_i\hbar)}{P_i(u)}$$

It follows from (ii) that $\lambda(\alpha_i^\vee) = \deg(P_i) \in \mathbb{N}$ and hence that λ is a dominant integral weight. The polynomials $\{P_i(u)\}$ are called Drinfeld polynomials. By Theorem 3.2, the set of isomorphism classes of simple objects in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ is in bijection with the set $\mathcal{P}_+^Y$ of pairs $(\lambda \in \mathfrak{h}^*, \{P_i(u) \in \mathbb{C}[u]\}_{i \in \mathbf{I}})$ such that $\lambda(\alpha_i^\vee) = \deg(P_i)$ and P_i are monic. If $(\lambda, \{P_i\}) \in \mathcal{P}_+^Y$, we denote by $L(\lambda, \{P_i\})$ the corresponding irreducible $Y_\hbar(\mathfrak{g})$–module.

3.3. Drinfeld polynomials: quantum loop algebra.

Let $\gamma = \{\gamma_{i,\pm m}^{\pm}\}_{i\in\mathbf{I}, m\in\mathbb{N}}$ be a collection of complex numbers and $\lambda \in \mathfrak{h}^*$ such that $\gamma_{i,0}^{\pm} = q_i^{\pm\lambda(\alpha_i^{\vee})}$. A representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ is said to be an l–highest weight representation of l-highest weight (λ, γ) if there exists $\mathbf{v} \in \mathcal{V}$ such that

(i) $\mathcal{V} = U_q(L\mathfrak{g})\mathbf{v}$.

(ii) $\mathcal{X}_{i,k}^{+}\mathbf{v} = 0$ for every $i \in \mathbf{I}$ and $k \in \mathbb{Z}$.

(iii) $\Psi_{i,\pm m}^{\pm}\mathbf{v} = \gamma_{i,\pm m}^{\pm}\mathbf{v}$ and $K_h\mathbf{v} = q^{\lambda(h)}\mathbf{v}$ for any $i \in \mathbf{I}, m \in \mathbb{N}$ and $h \in \mathfrak{h}$.

As for Yangians, for any (λ, γ), there is a unique irreducible representation with highest weight (λ, γ).

Theorem.

(i) *Every irreducible representation in $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ is a highest weight representation for a unique highest weight (λ, γ).*

(ii) *The irreducible representation $\mathcal{L}(\lambda, \gamma)$ is in $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ if, and only if there exist monic polynomials $\{\mathcal{P}_i(w) \in \mathbb{C}[w]\}_{i\in\mathbf{I}}$, $\mathcal{P}_i(0) \neq 0$, such that*

$$\sum_{m\geq 0} \gamma_{i,m}^{+}z^{-m} = q_i^{-\deg(\mathcal{P}_i)}\frac{\mathcal{P}_i(q^2 z)}{\mathcal{P}_i(z)} = \sum_{m\leq 0}\gamma_{i,m}^{-}z^m$$

Again, we have $\lambda(\alpha_i^{\vee}) = \deg(\mathcal{P}_i)$ and hence λ is a dominant integral weight. The polynomials $\mathcal{P}_i$ are again called Drinfeld polynomials. The set of isomorphism classes of simple objects in $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ is in bijection with the set $\mathcal{P}_{+}^{U}$ of pairs $(\lambda \in \mathfrak{h}^*, \{\mathcal{P}_i\})$ such that $\mathcal{P}_i$ are monic, $\mathcal{P}_i(0) \neq 0$ and $\lambda(\alpha_i^{\vee}) = \deg(\mathcal{P}_i)$. We denote by $\mathcal{L}(\lambda, \{\mathcal{P}_i\})$ the irreducible $U_q(L\mathfrak{g})$–module corresponding to $(\lambda, \{\mathcal{P}_i\}) \in \mathcal{P}_{+}^{U}$.

3.4. Composition series.

We shall need the analogue for $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ of the existence of composition series in category $\mathcal{O}$ for an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$, (see [36, Lemma 9.6] and [32, Prop. 15]).

Lemma.

(i) *Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ and $\lambda \in \mathfrak{h}^*$. Then, there exists a filtration of $U_q(L\mathfrak{g})$–modules*

$$0 = \mathcal{V}_0 \subset \cdots \subset \mathcal{V}_t = \mathcal{V}$$

such that the following holds for any $\mathcal{V}_j$, $j = 1, \ldots, t$

- *either $\mathcal{V}_j/\mathcal{V}_{j-1} = \mathcal{L}(\lambda_j, \{\mathcal{P}_i^{(j)}\})$, for some $\lambda_j \geq \lambda$,*
- *or $(\mathcal{V}_j/\mathcal{V}_{j-1})_\mu = 0$ for every $\mu \geq \lambda$.*

Moreover, given $\mu \geq \lambda$ and $(\mu, \mathcal{P}_i) \in \mathcal{P}_{+}^{U}$, the number of times $\mathcal{L}(\mu, \mathcal{P}_i)$ appears as $\mathcal{V}_j/\mathcal{V}_{j-1}$ is independent of λ and the filtration chosen, and is denoted by $[\mathcal{V} : \mathcal{L}(\mu, \mathcal{P}_i)]$.

(ii) *Let $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\lambda \in \mathfrak{h}^*$. Then, there exists a filtration of $Y_\hbar(\mathfrak{g})$–modules*

$$0 = V_0 \subset \cdots \subset V_t = V$$

such that the following holds for any $\mathcal{V}_j$, $j = 1, \ldots, t$

- *either $V_j/V_{j-1} = L(\lambda_j, \{P_i^{(j)}\})$ for some $\lambda_j \geq \lambda$.*
- *or $(V_j/V_{j-1})_\mu = 0$ for every $\mu \geq \lambda$.*

Moreover, given $\mu \geq \lambda$ and $(\mu, P_i) \in \mathcal{P}_+^Y$, the number of times $L(\mu, P_i)$ appears as V_j/V_{j-1} is independent of λ and the filtration chosen, and is denoted by $[V : L(\mu, P_i)]$.

3.5. Definition of the categories $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$.

Let $\Pi \subset \mathbb{C}$ be a subset such that $\Pi \pm \frac{\hbar}{2} \subset \Pi$. We define $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ to be the full subcategory of $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ consisting of the representations V such that, for every $(\lambda, \{P_i\}) \in \mathcal{P}_+^Y$ for which $[V : L(\lambda, \{P_i\})] \neq 0$, the roots of P_i lie in Π for every $i \in \mathbf{I}$.

Let similarly $\Omega \subset \mathbb{C}^\times$ be a subset stable under multiplication by $q^{\pm 1}$. We define $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$ to be the full subcategory of $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ consisting of those $\mathcal{V}$ such that every $(\lambda, \{\mathcal{P}_i\}) \in \mathcal{P}_+^U$ for which $[\mathcal{V} : \mathcal{L}(\lambda, \{\mathcal{P}_i\})] \neq 0$, the roots of $\mathcal{P}_i$ lie in Ω for every $i \in \mathbf{I}$.

Proposition.

(i) *$\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$ are Serre subcategories of $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ respectively. That is, they are closed under taking direct sums, subobjects, quotient and extensions.*

(ii) *When $\mathfrak{g}$ is a simple Lie algebra, $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$ are closed under tensor product.*

PROOF. The first part is obvious from the definition. The second part will follow from the alternate characterizations of these categories given in Theorem 3.8 (iv), since by [37, Lemma 1] and [24] $\xi_i(u)$ and $\Psi_i^\pm(z)$ are group–like modulo off–diagonal entries. $\square$

3.6. Rationality of fields.

The aim of this paragraph is to prove that the fields giving the action of the generators of $Y_\hbar(\mathfrak{g}), U_q(L\mathfrak{g})$ on category $\mathcal{O}$ integrable representations are rational functions. The statement for $U_q(L\mathfrak{g})$ is well known, see [5, §6] and [33, Proposition 3.8]. Our proof is similar, but yields in addition the explicit form of these rational functions.

Proposition.

(i) *Let V be a $Y_\hbar(\mathfrak{g})$–module on which $\mathfrak{h}$ acts semisimply with finite–dimensional weight spaces. Then, for every weight μ of V, the generating series*

$$\xi_i(u) \in \mathrm{End}(V_\mu)[[u^{-1}]] \qquad x_i^\pm(u) \in \mathrm{Hom}(V_\mu, V_{\mu \pm \alpha_i})[[u^{-1}]]$$

defined in 2.3 are the expansions at ∞ of rational functions of u. Specifically, let $t_{i,1} = \xi_{i,1} - \dfrac{\hbar}{2}\xi_{i,0}^2 \in Y_\hbar(\mathfrak{g})^\mathfrak{h}$. Then,

$$x_i^\pm(u) = \hbar u^{-1}\left(1 \mp \frac{\mathrm{ad}(t_{i,1})}{2d_i u}\right)^{-1} x_{i,0}^\pm$$

and $\xi_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$.

(ii) *Let $\mathcal{V}$ be a $U_q(L\mathfrak{g})$–module on which the operators K_h act semisimply with finite–dimensional weight spaces. Then, for every weight μ of $\mathcal{V}$ and $\varepsilon \in \{\pm\}$, the generating series*

$$\Psi_i(z)^\pm \in \mathrm{End}(\mathcal{V}_\mu)[[z^{\mp 1}]] \qquad \mathcal{X}_i^\varepsilon(z)^\pm \in \mathrm{Hom}(\mathcal{V}_\mu, \mathcal{V}_{\mu \pm \alpha_i}))[[z^{\mp}]]$$

defined in 2.10 are the expansions of rational functions $\Psi_i(z), \mathcal{X}_i^\varepsilon(z)$ at $z = \infty$ and $z = 0$. Specifically, let $H_{i,\pm 1} = \pm \Psi_{i,0}^\mp \Psi_{i,\pm 1}^\pm / (q_i - q_i^{-1})$. Then,

$$\mathcal{X}_i^\varepsilon(z) = \left(1 - \varepsilon \frac{\mathrm{ad}(H_{i,1})}{[2]_i z}\right)^{-1} \mathcal{X}_{i,0}^\varepsilon = -z\left(1 - \varepsilon z \frac{\mathrm{ad}(H_{i,-1})}{[2]_i}\right)^{-1} \mathcal{X}_{i,-1}^\varepsilon$$

and $\Psi_i(z) = \Psi_{i,0}^- + (q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_{i,0}^-]$.

PROOF. (i) The relations (Y2) and (Y3) imply that $[t_{i,1}, x_{i,r}^\pm] = \pm 2d_i x_{i,r+1}^\pm$, so that

$$[t_{i,1}, x_i^\pm(u)] = \pm 2d_i(ux_i^\pm(u) - \hbar x_{i,0})$$

The relation $\xi_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$ is a direct consequence of (Y5).

(ii) The relations (QL2) and (QL3) imply that $[H_{i,\pm 1}, \mathcal{X}_{i,k}^\varepsilon] = \varepsilon[2]_i \mathcal{X}_{i,k\pm 1}^\varepsilon$, and therefore that

$$\mathcal{X}_i^\varepsilon(z)^+ = \left(1 - \varepsilon \frac{\mathrm{ad}(H_{i,1})}{[2]_i z}\right)^{-1} \mathcal{X}_{i,0}^\varepsilon$$

$$\mathcal{X}_i^\varepsilon(z)^- = -z\left(1 - \varepsilon z \frac{\mathrm{ad}(H_{i,-1})}{[2]_i}\right)^{-1} \mathcal{X}_{i,-1}^\varepsilon$$

which are rational functions regular at $z = \infty$ and $z = 0$ respectively. To see that these functions are the same, it suffices to note that, as a formal power series, and therefore as a rational function, $\mathcal{X}_i^\varepsilon(z)^-$ also satisfies

$$\left(1 - \varepsilon \frac{\mathrm{ad}(H_{i,1}^+)}{[2]_i z}\right) \mathcal{X}_i^\varepsilon(z)^- = \mathcal{X}_{i,0}^\varepsilon$$

Finally, note that $\Psi_i^\pm(z) = \Psi_{i,0}^- + (q_i - q_i^{-1})[X_i^+(z)^\pm, \mathcal{X}_{i,0}^-]$.   □

3.7.   The following result will be needed elsewhere

Lemma.

(i) *Let $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, and let $\mu \in \mathfrak{h}^*$ be a weight of V. Then, the poles of $\xi_i(u)_\mu$ are contained in the union of the poles of $x_i^\pm(u)_\mu$ and those of $x_i^\pm(u)_{\mu \mp \alpha_i}$.*

(ii) *Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$, and let $\mu \in \mathfrak{h}^*$ be a weight of $\mathcal{V}$. Then the poles of $\Psi_i(z)_\mu$ are containes in the union of the poles of $\mathcal{X}_i^\pm(z)_\mu$ and those of $\mathcal{X}_i^\pm(z)_{\mu \mp \alpha_i}$.*

PROOF. (i) and (ii) follow from

$$\xi_i(u) = 1 + [x_i^+(u), x_{i,0}^-] = 1 + [x_{i,0}^+, x_i^-(u)]$$

$$\Psi_i(z) = \Psi_{i,0}^- + (q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_{i,0}^-] = \Psi_{i,0}^- + (q_i - q_i^{-1})[\mathcal{X}_{i,0}^+, \mathcal{X}_i^-(z)]$$

respectively   □

3.8. **Equivalent characterizations of $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$.** In view of Proposition 3.6, we can define, for a given representation (ρ, V) in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, a subset $\sigma(V) \subset \mathbb{C}$ consisting of the poles of the rational functions $\{\rho(\xi_i(u))^{\pm 1}, \rho(x_i^\pm(u))\}$. Similarly, for a representation $(\rho, \mathcal{V})$ in $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$, we define a subset $\sigma(\mathcal{V}) \subset \mathbb{C}^\times$ consisting of poles of the functions $\{\rho(\Psi_i(z))^{\pm 1}, \rho(\mathcal{X}_i^\pm(z))\}$.

Theorem.
   (1) *Let $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$. Then, the following conditions are equivalent.*
      (i) *$V \in \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$.*
      (ii) *$\sigma(V) \subset \Pi$.*
      (iii) *The poles of $\xi_i(u)^{\pm 1}$ are contained in Π.*
      (iv) *The eigenvalues of $\xi_i(u)$ have zeroes and poles in Π.*
   (2) *Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$. Then, the following conditions are equivalent.*
      (i) *$\mathcal{V} \in \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$.*
      (ii) *$\sigma(\mathcal{V}) \subset \Omega$.*
      (iii) *The poles of $\Psi_i(z)^{\pm 1}$ are contained in Ω.*
      (iv) *The eigenvalues of $\Psi_i(z)$ have zeroes and poles in Ω.*

PROOF. The rest of this section is devoted to the proof of Theorem 3.8 for the Yangian $Y_\hbar(\mathfrak{g})$. The assertion for $U_q(L\mathfrak{g})$ is proved similarly and we therefore omit it. The following implications are obvious: (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). We prove (i) $\Rightarrow$ (ii) by induction on the length of the composition series of V. More precisely, assume (i), and fix a weight $\mu \in \mathfrak{h}^*$. We need to prove that the operators $\{\xi_i(u)_\mu^{\pm 1}, x_i^\pm(u)_\mu\}_{i \in \mathbf{I}}$ have poles in Π. Fix an arbitrary $\lambda \in \mathfrak{h}^*$ such that $\lambda \leq \mu - \alpha_i$ for every $i \in \mathbf{I}$, and consider a composition series given by Lemma 3.4. The proof of (i) $\Rightarrow$ (ii) is by induction on the length of this composition series. Proposition 3.9 deals with the case of an irreducible V, while Proposition 3.10 carries out the induction step.   □

3.9.

Proposition. *Let V be an irreducible representation in $\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$. Then $V \in \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$ if, and only if $\sigma(V) \subset \Pi$.*

PROOF. It is clear that if $\sigma(V) \subset \Pi$ then the roots of $P_i(u)$ are in Π (since roots of $P_i(u)$ are obtained by successively shifting the poles of $\xi_i(u)$ acting on the highest weight vector by $d_i\hbar$, and Π is closed under these shifts). Hence, by definition, $V \in \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$.

Now we assume that the roots of $P_i(u)$ are in Π, for every $i \in \mathbf{I}$, and prove that all the rational functions $\{\xi_i(u), x_i^\pm(u)\}$ have poles in Π. For the purposes of the proof, we write the weight space decomposition of V as a $\mathfrak{g}$–module

$$V = \oplus_{\nu \in \mathfrak{h}^*} V_\nu$$

and for $y \in Y_\hbar(\mathfrak{g})$ we write y_ν for the operator it defines on the weight space V_ν.

Let $\mu \in \mathfrak{h}^*$ be the highest weight of V. Using the results of 3.2 we have the following properties of V, which will be used in the proof:

(P1) Every weight space V_ν, for $\nu < \mu$, is spanned by $\{x_{i,r}^- V_{\nu+\alpha_i}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$.

(P2) If $v \in V_\nu$ is annihilated by $x_{i,r}^+$ for every $i \in \mathbf{I}$ and $r \in \mathbb{N}$, and $\nu \neq \mu$, then $v = 0$.

The proof that each rational function $\xi_i(u)_\nu, x_i^\pm(u)_\nu$ has poles in Π is by induction on the height of $\mu - \nu$. Recall that for an element $\alpha = \sum_i n_i \alpha_i \in Q_+$, the height of α, denoted by $\mathrm{ht}(\alpha)$, is defined as:

$$\mathrm{ht}(\alpha) = \sum_i n_i \in \mathbb{N}$$

More precisely, we prove the following statement, by induction on $k \in \mathbb{N}$:

$\mathbf{S}(k)$: For every $i \in \mathbf{I}$, the rational functions $\xi_i(u)_\nu, x_i^+(u)_\nu$ have poles in Π, for every ν such that $\mathrm{ht}(\mu - \nu) \leq k$; and $x_i^-(u)_\nu$ has poles in Π for every ν such that $\mathrm{ht}(\mu - \nu) < k$.

The base case of $\mathbf{S}(0)$ is clear since $x_i^+(u)_\mu = 0$ and

$$\xi_i(u)_\mu = \frac{P_i(u + d_i \hbar)}{P_i(u)}$$

Let us assume $\mathbf{S}(k')$ for every $k' \leq k$, where $k \geq 0$, and prove $\mathbf{S}(k+1)$. Let ν be a weight of V such that $\mathrm{ht}(\mu - \nu) = k + 1$ and let $i, j \in \mathbf{I}$. Using the relation $(\mathcal{Y}5)$ we have:

$$x_i^+(u)_\nu x_j^-(v)_{\nu+\alpha_j} = x_j^-(v)_{\nu+\alpha_i+\alpha_j} x_i^+(u)_{\nu+\alpha_j} + \frac{\delta_{ij}\hbar}{u-v}\left(\xi_i(v)_{\nu+\alpha_i} - \xi_i(u)_{\nu+\alpha_i}\right)$$
$$(3.1)$$

Note that the right–hand side has poles in $\Pi \times \Pi$ by the induction hypothesis. This allows us to conclude the same for $x_i^+(u)_\nu$ and $x_j^-(v)_{\nu+\alpha_j}$ as follows. Assume that $x_j^-(v)_{\nu+\alpha_j}$ has a pole at $z \in \mathbb{C} \setminus \Pi$ of order n. Multiplying both sides of the equation (3.1) by $(v - z)^n$ and setting $v = z$, we get:

$$x_i^+(u)_\nu \left((v - z)^n x_j^-(v)_{\nu+\alpha_j}\Big|_{v=z} \right) = 0$$

Thus the image of the operator $\left((v-z)^n x_j^-(v)_{\nu+\alpha_j}\big|_{v=z}\right)$ is annihilated by all $x_{i,r}^+$, which implies that this operator is zero, since $\nu \neq \mu$ (see property (P2) above). This is a contradiction to the fact that z is a pole of order n of $x_j^-(v)_{\nu+\alpha_j}$. The proof for $x_i^+(u)_\nu$ is similar.

It remains to show that $\xi_i(u)_\nu$ has poles in Π. For this we use the relation $(\mathcal{Y}3)$

$$\xi_i(u)_\nu x_j^-(v)_{\nu+\alpha_j} = \frac{u-v-a}{u-v+a} x_j^-(v)_{\nu+\alpha_j} \xi_i(u)_{\nu+\alpha_j} + \frac{2a}{u-v+a} x_j^-(u+a)_{\nu+\alpha_j} \xi_i(u)_{\nu+\alpha_j}$$

where $a = \hbar d_i a_{ij}/2$.

This relation implies that the poles of $\xi_i(u)$ acting on the image of the operator $x_j^-(v)_{\nu+\alpha_j}$ in V_ν are either the poles of $\xi_i(u)_{\nu+\alpha_j}$ or $x_j^-(u+a)_{\nu+\alpha_j}$, which are in Π, by the induction hypothesis and the fact that Π is stable under shift by a. Since V_ν is spanned by such subspaces (property (P1)), we are done. $\qquad\square$

3.10.

Proposition. *Let* $0 \to V_1 \to V \to V_2 \to 0$ *be a short exact sequence of integrable* $Y_\hbar(\mathfrak{g})$*–modules. If* $\sigma(V_1), \sigma(V_2) \subset \Pi$ *then* $\sigma(V) \subset \Pi$.

PROOF. Let us fix $\mu \in \mathfrak{h}^*$ and $i \in \mathbf{I}$. We will prove that for every $k \in \mathbb{Z}$ the poles of $\xi_i(u)_{\mu+k\alpha_i}, x_i^\pm(u)_{\mu+k\alpha_i}$ acting on the corresponding weight space of V are in Π.

We write $V = V_1 \oplus V_2$ as vector spaces. An element $y \in Y_\hbar(\mathfrak{g})$ has the following form (viewed as an element of $\mathrm{End}(V)$):

$$y = \begin{bmatrix} y^{11} & y^{12} \\ 0 & y^{22} \end{bmatrix}$$

By assumption, $\xi_i(u)^{kk}, x_i^\pm(u)^{kk}$ have poles in Π, for $k = 1, 2$. Let us assume that $z \notin \Pi$ is a pole of one of the functions $\xi_i(u)_\nu^{12}, x_i^\pm(u)_\nu^{12}$ for $\nu \in \mu + \mathbb{Z}\alpha_i$. Let N be the maximum of the order of the pole at z of these functions (note that by definition, there are only finitely many $k \in \mathbb{Z}$ such that $V_{\mu+k\alpha_i} \neq 0$). Define for every $\nu \in \mu + \mathbb{Z}\alpha_i$

$$\mathsf{H}_\nu = \lim_{u \to z} (u-z)^N \xi_i(u)_\nu^{12}$$

$$\mathsf{X}_\nu^\pm = \lim_{u \to z} (u-z)^N x_i^\pm(u)_\nu^{12}$$

Using the relation $(\mathcal{Y}5)$ and taking its $(1,2)$ entry, we have:

$$(u-v)(x_i^+(u)_{\nu-\alpha_i}^{11} x_i^-(v)_\nu^{12} + x_i^+(u)_{\nu-\alpha_i}^{12} x_i^-(v)_\nu^{22}$$
$$- x_i^-(v)_{\nu+\alpha_i}^{11} x_i^+(u)_\nu^{12} - x_i^-(v)_{\nu+\alpha_i}^{12} x_i^+(u)_\nu^{22}) = \hbar(\xi_i(v)_\nu^{12} - \xi_i(u)_\nu^{12})$$

Multiplying both sides by $(u - z)^N$ and letting $u \to z$ we get

$$(z - v)(\mathsf{X}^+_{\nu-\alpha_i} x^-_i(v)^{22}_\nu - x^-_i(v)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu) = -\hbar \mathsf{H}_\nu$$

and similarly (using $v - z$ instead of $u - z$) we have

$$(u - z)(x^+_i(u)^{11}_{\nu-\alpha_i} \mathsf{X}^-_\nu - \mathsf{X}^-_{\nu+\alpha_i} x^+_i(u)^{22}_\nu) = \hbar \mathsf{H}_\nu$$

These equations clearly imply that $\mathsf{H}_\nu = 0$ and we have the following relations:

$$\mathsf{X}^+_{\nu-\alpha_i} x^-_i(v)^{22}_\nu = x^-_i(v)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu \tag{3.2}$$

$$x^+_i(u)^{11}_{\nu-\alpha_i} \mathsf{X}^-_\nu = \mathsf{X}^-_{\nu+\alpha_i} x^+_i(u)^{22}_\nu \tag{3.3}$$

Now we consider the relation $(\mathcal{Y}3)$ in the following form

$$(u - v - d_i\hbar)\xi_i(u)x^+_i(v) - (u - v + d_i\hbar)x^+_i(v)\xi_i(u)$$
$$= -d_i\hbar(\xi_i(u)x^+_i(u) + x^+_i(u)\xi_i(u))$$

Again taking its $(1, 2)$ component, multiplying with $(u - z)^N$ and letting $u \to z$, and using the fact that $\mathsf{H} = 0$ we get:

$$\xi_i(z)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu + \mathsf{X}^+_\nu \xi_i(z)^{22}_\nu = 0 \tag{3.4}$$

Similar computation with relation $(\mathcal{Y}4)$ yields

$$(z - v - d_i\hbar)\mathsf{X}^+_{\nu+\alpha_i} x^+_i(v)^{22}_\nu - (z - v + d_i\hbar)x^+_i(v)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu$$
$$= -d_i\hbar(x^+_i(z)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu + \mathsf{X}^+_{\nu+\alpha_i} x^+_i(z)^{22}_\nu) \tag{3.5}$$

We take commutator of (3.5) with $x^-_{i,0}$, using the commutativity property (3.2), the equation (3.4) and the relation $[x^+_i(u), x^-_{i,0}] = \xi_i(u) - 1$, to obtain:

$$(z - v - d_i\hbar)\mathsf{X}^+_\nu \xi_i(v)^{22}_\nu - (z - v + d_i\hbar)\xi_i(v)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu = 0$$

If we set $v = z - d_i\hbar$ in this equation, we get

$$-2d_i\hbar\xi_i(z - d_i\hbar)^{11}_{\nu+\alpha_i} \mathsf{X}^+_\nu = 0$$

But since $z - d_i\hbar \notin \Pi$ and $\xi_i(u)^{11}$ is invertible there, we obtain that $\mathsf{X}^+_\nu = 0$. A similar argument shows that $\mathsf{X}^-_\nu = 0$, which contradicts the fact that N was the order of the pole at z of one of $\xi_i(u)^{12}_\nu, x^\pm_i(u)^{12}_\nu$. This contradiction proves that all the poles are in Π and we are done. $\qquad\square$

4. ADDITIVE DIFFERENCE EQUATIONS

In this section, we give a self–contained account of Birkhoff's theory of additive difference equations, including a detailed discussion of the inverse problem for abelian ones. Our exposition, and understanding of the subject owes much to [6, 8, 38]. Additional information may be found in [7, 51].

4.1.   Let V be a finite–dimensional vector space over $\mathbb{C}$, and let $A : \mathbb{C} \to \mathrm{End}(V)$ be a rational function. We assume that A is regular at ∞ and such that $A(\infty) = 1$. Consider the following system of difference equations for a function $\phi : \mathbb{C} \to \mathrm{End}(V)$

$$\phi(u + 1) = A(u)\phi(u) \tag{4.1}$$

Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be the subalgebra generated by $A(u)$, $u \in \mathbb{C}$, or equivalently by the coefficients of the Taylor expansion

$$A(u) = 1 + A_0 u^{-1} + A_1 u^{-2} + \cdots$$

of A near ∞. The system (4.1) is said to be *non–resonant* relative to a subalgebra $\widetilde{\mathcal{A}} \subseteq \mathrm{End}(V)$ containing $\mathcal{A}$ if the eigenvalues of $\mathrm{ad}(A_0)$ on $\widetilde{A}$ do not lie in $\mathbb{Z}^\times$. Note that this is the case in either of the following situations:

- The eigenvalues of A_0 do not differ by positive integers, and $\widetilde{\mathcal{A}} = \mathrm{End}(V)$.
- The system (4.1) is abelian, that is $[A(u), A(v)] = 0$ for any $u, v \in \mathbb{C}$, and $\widetilde{\mathcal{A}} = \mathcal{A}$.

If (4.1) is non–resonant relative to $\widetilde{\mathcal{A}}$, it is easy to see that it possesses a unique formal solution of the form $\Upsilon(u)u^{A_0}$, where[6]

$$\Upsilon(u) = 1 + \Upsilon_0 u^{-1} + \Upsilon_1 u^{-2} + \cdots \in \widetilde{\mathcal{A}}[[u^{-1}]] \tag{4.2}$$

4.2. Canonical fundamental solutions.

Theorem. [6] *If the system (4.1) is non–resonant relative to $\widetilde{\mathcal{A}} \subseteq \mathrm{End}(V)$, there exist unique meromorphic solutions $\phi^\pm : \mathbb{C} \to \widetilde{\mathcal{A}}$ such that*

- (i) *$\phi^\pm$ is holomorphic and invertible for $\pm \mathrm{Re}(u) \gg 0$.*
- (ii) *$\phi^\pm$ possesses an asymptotic expansion of the form*

$$\phi^\pm(u) \sim (1 + H_0^\pm u^{-1} + H_1^\pm u^{-2} + \cdots) \cdot (\pm u)^{A_0} \tag{4.3}$$

  in any right (resp. left) half–plane, where $v^{A_0} = \exp(A_0 \log(v))$ is defined by choosing the standard determination of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Moreover,

- (iii) *The poles of $(\phi^-)^{\pm 1}$ are contained in $\mathcal{P} + \mathbb{N}^\times$ and $\mathcal{Z} + \mathbb{N}^\times$ respectively, where $\mathcal{P}, \mathcal{Z}$ are the poles of $A(u), A(u)^{-1}$.*
- (iv) *The poles of $(\phi^+)^{\pm 1}$ are contained in $\mathcal{Z} - \mathbb{N}$ and $\mathcal{P} - \mathbb{N}$ respectively.*
- (v) *The asymptotics (4.3) of $\phi^\pm$ are given by $\Upsilon(u)(\pm u)^{A_0}$.*

---

[6]For the formal solution $\Upsilon(u)u^{A_0}$, the equation (4.1) is understood to mean $\Upsilon(u+1) = A(u)\Upsilon(u)(1 + 1/u)^{-A_0}$, where $(1 + 1/u)^{-A_0} = \sum_{r \geq 0}(-1)^r \frac{A_0(A_0+1)\cdots(A_0+r-1)}{r!} u^{-r}$.

PROOF. The uniqueness of $\phi^{\pm}$ is proved in 4.3. In 4.4–4.6, we give a proof of the existence of the fundamental solutions $\phi^{\pm}$ under the simplifying assumption that A_0 commutes with $A(u)$ for every u. The general case is treated in [6, 38].

(iii)–(iv) If $\phi^-(u)$ is holomorphic and invertible for $\mathrm{Re}(u) < a$, then, for $\mathrm{Re}(u) < a + n$, $\phi^-(u) = A(u-1)\cdots A(u-n)\phi^-(u-n)$ (resp. $(\phi^-(u))^{-1} = \phi^-(u-n)^{-1}A(u-n)^{-1}\cdots A(u-1)^{-1}$), which has poles only if one of $u - 1, \ldots, u - n$ is a pole of $A(u)$ (resp. of $A(u)^{-1}$). Similarly for ϕ^+.

(v) Let $H^{\pm}(u) \in 1 + u^{-1}\widetilde{\mathcal{A}}[[u^{-1}]]$ be such that $\phi^{\pm}(\pm u)^{-A_0} \sim H^{\pm}(u)$ for $\pm \mathrm{Re}(u) > a$. Multiplying both sides of $\phi^{\pm}(u + 1) = A(u)\phi^{\pm}(u)$ by $(\pm(u+1))^{-A_0}$, using the fact that $(\pm u)^{A_0}(\pm(u+1))^{-A_0} = (1+1/u)^{-A_0}$ for $u \notin \mathbb{R}_{\leq 0}$ (resp. $u \notin \mathbb{R}_{\geq -1}$) and taking asymptotic expansions, shows that $H^{\pm}(u+1) = A(u)H^{\pm}(u)(1+1/u)^{-A_0}$, so that $H^{\pm}(u) = \Upsilon(u)$ by uniqueness. $\qquad\square$

4.3. Uniqueness of fundamental solutions.

Let $\phi_1^+(u), \phi_2^+(u)$ be two solutions of (4.1) satisfying the conditions (i)–(ii) of Theorem 4.2, and set $C(u) = \left(\phi_1^+(u)\right)^{-1}\phi_2^+(u)$. $C(u)$ is 1–periodic, and holomorphic for $\mathrm{Re}(u) \gg 0$ and therefore on the entire complex plane. As a holomorphic function of $z = e^{2\pi\iota u} = e^{2\pi\iota\,\mathrm{Re}(u)}e^{-2\pi\,\mathrm{Im}(u)}$, C has removable singularities at $z = 0$ and $z = \infty$. Indeed,

$$C(u) = u^{-A_0}\left(\phi_1^+(u)u^{-A_0}\right)^{-1}\left(\phi_2^+(u)u^{-A_0}\right)u^{A_0} \qquad (4.4)$$

Since C is 1–periodic, we may assume that $\mathrm{Re}\,u \gg 0$, so that the second and third factors tend to 1 as $\mathrm{Im}\,u \to \pm\infty$. Since the first and last ones grow like a polynomial in u, it follows that $\lim_{z\to 0} zC(z) = 0 = \lim_{z\to\infty} z^{-1}C(z)$. Thus, $C(u) \equiv C$ is a constant element of $\widetilde{\mathcal{A}}$. By (4.4),

$$u^{A_0}Cu^{-A_0} \sim 1 + C^0 u^{-1} + \cdots$$

for $\mathrm{Re}\,u \gg 0$. Let $A_0 = S_0 + N_0$ be the Jordan decomposition of A_0, and $C = \sum_{\lambda\in\mathbb{C}} C_\lambda$ the decomposition of C into eigenvectors of $\mathrm{ad}(S_0)$. Since

$$\mathrm{Ad}(u^{A_0})C_\lambda = u^\lambda \sum_{k\geq 0} \log(u)^k \frac{\mathrm{ad}(N_0)^k}{k!} C_\lambda$$

it follows that each C_λ is an eigenvector of $\mathrm{ad}(A_0)$, that $C_\lambda = 0$ for any $\lambda \notin \mathbb{Z}_{\leq 0}$, and that $C_0 = 1$. The non–resonance condition then implies that $C_n = 0$ for any $n \in \mathbb{Z}_{<0}$, so that $C = 1$. The uniqueness of ϕ^- follows in the same way.

4.4. Existence of fundamental solutions, I.

Assume first that $A_0 = 0$. The system (4.1) is then called *regular*, and its fundamental solutions are

given by

$$\phi^+(u) = \prod_{n\geq 0}^{\rightarrow} A(u+n)^{-1} = A(u)^{-1}A(u+1)^{-1}\cdots$$

$$\phi^-(u) = \prod_{n\geq 1}^{\rightarrow} A(u-n) = A(u-1)A(u-2)\cdots$$

It is easy to prove that the above products converge locally uniformly on the complement of $\mathcal{Z} - \mathbb{N}$, $\mathcal{P} + \mathbb{N}^\times$ respectively, and have the required asymptotics.

4.5. **Holomorphic functional calculus** [17]**.** Recall that given $X \in \mathrm{End}(V)$, and a meromorphic function f which is holomorphic on a neighborhood of the eigenvalues of X, $f(X) \in \mathrm{End}(V)$ is defined by the Cauchy integral

$$f(X) = \frac{1}{2\pi\iota}\int_C f(v)(v-X)^{-1}\,dv$$

where C is any contour enclosing all of the eigenvalues of X and none of the poles of $f(v)$. If X is semisimple, with eigenvalues $\lambda_1,\ldots,\lambda_k$ and corresponding eigenspaces $V_1,\ldots,V_k$, the residue theorem shows that $f(X)$ is the semisimple endomorphism acting as multiplication by $f(\lambda_i)$ on V_i. More generally, if $X = X_S + X_N$ is the Jordan decomposition of X, expanding $(v-X)^{-1}$ as

$$(v-X)^{-1} = \sum_{k\geq 0}(v-X_S)^{-k-1}X_N^k$$

a finite sum since X_N is nilpotent, shows that

$$f(X) = \sum_{k\geq 0}\frac{f^{(k)}(X_S)}{k!}X_N^k \tag{4.5}$$

If $f(u) = \sum_{i=0}^n a_i u^i$ is a polynomial, then $f(X) = \sum_i a_i X^i$. Moreover, $f_1 \cdot f_2(X) = f_1(X)f_2(X)$ when both sides are defined. Finally, if f_n is sequence converging locally uniformly to f, then $f_n(X)$ converges to $f(X)$. It follows for example that if

$$\Gamma(v) = v^{-1}e^{-\gamma v}\prod_{n\geq 1}\left(1+\frac{v}{n}\right)^{-1}e^{v/n}$$

is the Euler Gamma function, where γ is the Euler–Mascheroni constant, then

$$\Gamma(X) = X^{-1}e^{-\gamma X}\prod_{n\geq 1}\left(1+\frac{X}{n}\right)^{-1}e^{X/n} \tag{4.6}$$

4.6. **Existence of fundamental solutions, II.** Assume now that A_0 commutes with $A(u)$. We may then regularize (4.1) as follows. Set

$$\overline{A}(u) = (1 - A_0 u^{-1})A(u)$$

The difference equation $\overline{\phi}(u+1) = \overline{A}(u)\overline{\phi}(u)$ is regular, and its fundamental solutions $\overline{\phi}^{\pm}$ can be constructed as in 4.4. The fundamental solutions to (4.1) are $\phi^{\pm}(u) = f^{\pm}(u)\overline{\phi}^{\pm}(u)$, where $f^{\pm}(u)$ are the solutions of $f(u+1) = u(u - A_0)^{-1}f(u)$ given by

$$f^+(u) = \Gamma(u)\Gamma(u - A_0)^{-1}$$
$$f^-(u) = \Gamma(1 - u + A_0)\Gamma(1 - u)^{-1}$$

We will show in 4.7 that $f^{\pm}(u)$ have the required asymptotics as $u \to \infty$. Assuming this for now, and using (4.6), we can rewrite $\phi^{\pm}$ as

$$\phi^+(u) = e^{-\gamma A_0} A(u)^{-1} \overset{\rightarrow}{\prod_{n \geq 1}} A(u + n)^{-1} e^{A_0/n} \qquad (4.7)$$

$$\phi^-(u) = e^{-\gamma A_0} \overset{\rightarrow}{\prod_{n \geq 1}} A(u - n)\, e^{A_0/n} \qquad (4.8)$$

4.7. **Asymptotics of $f^{\pm}$.** Recall Sterling's asymptotic expansion [53, §12.33]

$$\Gamma(u) \sim e^{-u} u^{u-1/2} \sqrt{2\pi} \left(1 + 1/12u + 1/288u^2 + \cdots\right)$$

valid on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. This implies that, for any $\lambda \in \mathbb{C}$

$$\frac{\Gamma(u - \lambda)}{\Gamma(u)} \sim u^{-\lambda}(1 + h_1 u^{-1} + \cdots)$$

and, differentiating with respect to λ, that, for any $k \geq 0$

$$\frac{\Gamma^{(k)}(u)}{\Gamma(u)} \sim (\log(u))^k (1 + h_1^{(k)} u^{-1} + \cdots)$$

Let now $A_0 = S_0 + N_0$ be the Jordan decomposition of A_0. By (4.5),

$$
\begin{aligned}
f^+(u)^{-1} &= \Gamma(u - A_0)\Gamma(u)^{-1} \\
&= \frac{\Gamma(u - S)}{\Gamma(u)} \sum_{k \geq 0} \frac{\Gamma^{(k)}(u - S)}{\Gamma(u - S)} \frac{(-N)^k}{k!} \\
&\sim u^{-A_0}(1 + \cdots)
\end{aligned}
$$

Similarly, $f^-(u) \sim (1 + \cdots)(-u)^{A_0}$.

4.8. **Connection Matrix.** Define the *connection matrix* of the system (4.1) to be the function $S : \mathbb{C} \to GL(V)$ given by

$$S(u) = \left(\phi^+(u)\right)^{-1} \phi^-(u)$$

Clearly, $S(u)$ is 1–periodic, and therefore a function of $z = e^{2\pi\iota u}$.

Proposition. *S is a rational function of z, regular at $0, \infty$, and such that*

$$S(\infty) = e^{\pi\iota A_0} \qquad and \qquad S(0) = e^{-\pi\iota A_0}$$

PROOF. By Theorem 4.2, (iii)–(iv) S is a meromorphic function with finitely many poles in the z–plane. To determine its behaviour as $z \to \infty$ (resp. $z \to 0$), it suffices to consider the limit of $S(u)$ as $\mathrm{Im}(u) \to \mp\infty$, with $\mathrm{Re}\,u$ lying in an interval of the form $[a, a + 1[$. Since both asymptotic expansions $\phi^\pm \sim \Upsilon(u)(\pm u)^{A_0}$ hold on the strip $\mathrm{Re}(u) \in [a, a+1[$, we get

$$S(u) = u^{-A_0} \left(\phi^+(u)u^{-A_0}\right)^{-1} \left(\phi^-(u)(-u)^{-A_0}\right)(-u)^{A_0}$$
$$\sim u^{-A_0} \Upsilon(u)^{-1} \Upsilon(u)^{-1} (-u)^{A_0}$$
$$= u^{-A_0}(-u)^{A_0}$$

which is equal to $\exp(\pm\pi\iota A_0)$ depending on whether $\mathrm{Im}(u) \lessgtr 0$. $\qquad\square$

Note that, if $[A_0, A(u)] = 0$, it follows by (4.7)–(4.8) that

$$S(z) = \overset{\leftarrow}{\prod_{n \geq 1}} A(u + n)e^{-A_0/n}\, A(u) \overset{\rightarrow}{\prod_{n \geq 1}} A(u - n)e^{A_0/n}$$

$$= \cdots A(u + 2)A(u + 1)\, A(u)\, A(u - 1)A(u - 2)\cdots$$

where the last expression is understood as $\lim\limits_{n \to \infty} A(u + n) \cdots A(u + 1)A(u)$ $A(u - 1) \cdots A(u - n)$.

4.9. **Example.** The following basic example illustrates the constructions given above, and plays an important role in the computations below. Consider the scalar difference equation

$$\phi(u + 1) = \frac{u - a}{u - b}\phi(u)$$

with coefficient matrix $A(u) = \dfrac{u - a}{u - b}$. The canonical fundamental solutions $\phi^\pm$ are given by

$$\phi^+(u)\frac{\Gamma(u - a)}{\Gamma(u - b)} \qquad and \qquad \phi^-(u) = \frac{\Gamma(1 - u + b)}{\Gamma(1 - u + a)}$$

The connection matrix $S(u)$ is therefore equal to

$$S(u) = \frac{e^{2\pi\iota u} - e^{2\pi\iota a}}{e^{2\pi\iota u} - e^{2\pi\iota b}} \cdot e^{\pi\iota(b-a)}$$

where we used [53, §12.14]

$$\Gamma(u)\Gamma(1 - u) = \pi / \sin(\pi u) \qquad\qquad\qquad (4.9)$$

4.10. **Inverse problem.** Fix now $A_0 \in \text{End}(V)$, and let S be a rational function of $z = e^{2\pi\iota u}$ such that $S(\infty) = e^{\pi\iota A_0}$ and $S(0) = e^{-\pi\iota A_0}$. We now discuss the problem of finding a coefficient matrix $A(u)$ of the form $1 + A_0 u^{-1} + \cdots$, and such that $S(z)$ is the connection matrix of the system (4.1) determined by $A(u)$.

This may be phrased as the following factorization problem: find two meromorphic functions $\phi^{\pm}(u) : \mathbb{C} \to \text{End}(V)$ such that

(a) $\phi^{\pm}(u)$ are holomorphic and invertible for $\pm \text{Re}(u) \gg 0$.

(b) $\phi^{\pm}$ possesses an asymptotic expansion of the form

$$\phi^{\pm}(u) \sim (1 + H_0^{\pm} u^{-1} + H_1^{\pm} u^{-2} + \cdots) \cdot (\pm u)^{A_0}$$

valid in any right (resp. left) half–plane.

(c) $S(e^{2\pi\iota u}) = \phi^{+}(u)^{-1}\phi^{-}(u)$.

Once such a factorization is found, the coefficient matrix can be reconstructed as

$$A(u) = \phi^{-}(u+1)\phi^{-}(u)^{-1} = \phi^{+}(u+1)\phi^{+}(u)^{-1}$$

which is a rational function of u since its poles are contained in a strip $|\text{Re}\, u| < a$ by (a), and it tends to 1 as $u \to \infty$ by (b). By uniqueness, the canonical fundamental solutions of the difference equation defined by $A(u)$ are $\phi^{\pm}$, so that its connection matrix is S by (c).

The above factorization, and therefore the inverse problem do not always admit a solution. We refer the reader to [38] for several interesting cases when a solution can be obtained. In 4.14–4.16, we shall restrict our attention to the abelian case, that is assume that $[S(z), S(w)] = 0$. In this case a solution always exists, and can be written explicitly in terms of Gamma functions.

4.11. **Uniqueness of the factorization.** Assume that $\phi_1^{\pm}, \phi_2^{\pm}(u)$ satisfy the properties (a)–(c) of 4.10, and set

$$G(u) = \phi_2^{+}(u)\phi_1^{+}(u)^{-1} = \phi_2^{-}(u)\phi_1^{-}(u)^{-1}$$

G is meromorphic and, by (b) tends to 1 as $u \to \infty$, so that G is a rational function. Thus, $\phi_2^{\pm}(u) = G(u)\phi_1^{\pm}(u)$, and the corresponding coefficient matrices are related by the *isomonodromic transformation*

$$A_{\phi_2}(u) = \phi_2^{\pm}(u+1)\phi_2^{\pm}(u)^{-1} = G(u+1)A_{\phi_1}(u)G(u)^{-1}$$

The following gives a necessary condition for the uniqueness of A. Call two subsets $A, B \subset \mathbb{C}$ *non–congruent* if $a - b \notin \mathbb{Z}^{\times}$ for any $a \in A$, $b \in B$. We shall also say that A is non–congruent if it is non–congruent to itself.

Proposition. *Let $A(u), A'(u) : \mathbb{C} \to GL(V)$ be rational functions of the form $1 + A_0 u^{-1} + \cdots$, and assume that the connection matrices of the difference equations determined by A and A' are equal. Let $\mathcal{P}, \mathcal{Z} \subset \mathbb{C}$ (resp. $\mathcal{P}', \mathcal{Z}'$) be the poles of $A(u), A(u)^{-1}$ (resp. $A'(u), A'(u)^{-1}$). If the pairs*

$$(\mathcal{Z}, \mathcal{P}), \quad (\mathcal{Z}', \mathcal{P}'), \quad (\mathcal{Z}, \mathcal{Z}'), \quad (\mathcal{P}, \mathcal{P}')$$

are non–congruent, then $A = A'$.

PROOF. By the foregoing, there exists a rational function $G(u) : \mathbb{C} \to GL(V)$ with $G(\infty) = 1$, and such that $A'(u) = G(u+1)A(u)G(u)^{-1}$. Thus, $G(u+1) = A'(u)G(u)A(u)^{-1}$ so that, for any $n \in \mathbb{N}^{\times}$,

$$G(u) = A'(u-1) \cdots A'(u-n)\, G(u-n)\, A(u-n)^{-1} \cdots A(u-1)^{-1}$$

Since $G(u)$ is regular for $\mathrm{Re}(u) << 0$, this implies that the poles of G are contained in $(\mathcal{P}' \cup \mathcal{Z}) + \mathbb{N}^{\times}$. Similarly, $G(u) = A'(u)^{-1}G(u+1)A(u)$, so that for any $n \in \mathbb{N}$

$$G(u) = A'(u)^{-1} \cdots A'(u+n-1)^{-1}\, G(u+n)\, A(u+n-1) \cdots A(u)$$

implying that the poles of G are contained in $(\mathcal{Z}' \cup \mathcal{P}) - \mathbb{N}$. The non–congruence assumption then implies that G has no poles, hence $G \equiv 1$ and $A' = A$ as claimed $\qquad\square$

4.12.

Lemma. *Let $M : \mathbb{C} \to GL(V)$ be a rational function such that*

$$[M(u), M(v)] = 0$$

for any $u, v \in \mathbb{C}$. Then, the semisimple and unipotent components $M_S, M_U :$ $\mathbb{C} \to GL(V)$ of the Jordan decomposition of M are rational functions.

PROOF. Let $\mathfrak{a} \subset \mathrm{End}(V)$ be the span of $\{M(u)\,|\, u \in \mathbb{C}\}$. By assumption, $\mathfrak{a}$ is an abelian Lie subalgebra of $\mathrm{End}(V)$, so that $V = \bigoplus_{\lambda \in \mathfrak{a}^*} V_\lambda$, where V_λ is the generalized eigenspace

$$V_\lambda = \{v \in V\,|\, (x - \lambda(x))^N v = 0, \ \forall x \in \mathfrak{a}, N \gg 0\}$$

Set $M_S'(u) = \sum_\lambda \mathbf{1}_\lambda \circ \lambda(M(u))$, where $\mathbf{1}_\lambda : V \to V_\lambda$ is the projection operator, and $M_U'(u) = M_S'(u)^{-1}M(u)$. By construction, $M_S'(u), M_U'(u)$ commute, are semisimple and unipotent respectively, and satisfy $M(u) = M_S'(u)M_U'(u)$. By uniqueness of the Jordan decomposition $M_S = M_S'$ and $M_U = M_U'$ and the conclusion follows since M_S' and M_U' are rational functions. $\qquad\square$

4.13. **Poles of abelian matrix functions.** Retain the notation of §4.12, and let $\mathcal{P}, \mathcal{P}_S, \mathcal{P}_U$ be the sets of poles of $M(z), M_S(z), M_U(z)$, and $\mathcal{Z}, \mathcal{Z}_S, \mathcal{Z}_U$ those of $M(z)^{-1}, M_S(z)^{-1}, M_U(z)^{-1}$ respectively.

Lemma. *The following holds*
   (i) *$\mathcal{P} = \mathcal{P}_S \cup \mathcal{P}_U$ and $\mathcal{Z} = \mathcal{Z}_S \cup \mathcal{Z}_U$.*
   (ii) *$\mathcal{P}_U = \mathcal{Z}_U$.*
   (iii) *The eigenvalues of $M(z)$ are rational functions of z, and $\mathcal{P}_S$ (resp. $\mathcal{Z}_S$) consist of the poles (resp. zeros) of those eigenvalues.*

PROOF. (ii) follows from $M_U(z)^{\pm 1} = \exp(\pm m_U(z))$, with $m_U = \log(M_U(z))$. (i) and (iii) follow by choosing a basis of V in which $M_S(z)$ are all diagonal and $M_U(z)$ strictly upper triangular. $\qquad\square$

4.14. Fix $A_0 \in \mathrm{End}(V)$, and let $S : \mathbb{C} \to GL(V)$ be a rational function such that

- $S(\infty) = \exp(\pi\iota A_0) = S(0)^{-1}$
- $[S(z), S(w)] = 0$
- $[A_0, S(z)] = 0$

To factorise $S(z)$, we shall need to choose logarithms of its poles and zeros. Specifically, apply §4.13 to $M(z) = S(z)$ and set

$$\mathcal{Z}_S = \{\alpha_i\}_{i\in I} \qquad \mathcal{P}_S = \{\beta_j\}_{j\in J} \qquad \mathcal{P}_U = \{\delta_k\}_{k\in K} = \mathcal{Z}_U$$

Let $(\lambda(z), \mu)$ be a joint eigenvalue of $(S(z), A_0)$. Then,

$$\lambda(z) = e^{\pi\iota\mu} \prod_{i\in I', j\in J'} (z - \alpha_i)^{n_i}/(z - \beta_j)^{m_j}$$

for some $I' \subset I, J' \subset J$, and $n_i, m_j \in \mathbb{N}^\times$. Since $S(\infty) = S(0)^{-1}$, we get $\prod \beta_j^{m_j}\alpha_i^{-n_i} = e^{2\pi\iota\mu}$. Choose complex numbers $\{a_i^{(\lambda,\mu)}\}_{i\in I'}$ and $\{b_j^{(\lambda,\mu)}\}_{j\in J'}$ such that $e^{2\pi\iota a_i^{(\lambda,\mu)}} = \alpha_i$, $e^{2\pi\iota b_j^{(\lambda,\mu)}} = \beta_j$, and

$$\sum_j m_j b_j^{(\lambda,\mu)} - \sum_i n_i a_i^{(\lambda,\mu)} = \mu \tag{4.10}$$

Let $\{\delta_k\}_{k\in K'}$ be the poles of the restriction of $S_U(z)$ to the generalised eigenspace of $S(z)$ corresponding to (λ, μ), and choose $\{d_k^{(\lambda,\mu)}\}_{k\in \widetilde{K}}$ such that $e^{2\pi\iota d_k^{(\lambda,\mu)}} = \delta_k$. Finally, set

$$\{a_i\}_{i\in \widetilde{I}} = \bigcup_{(\lambda,\mu)} \{a_i^{(\lambda,\mu)}\}_{i\in I'} \qquad \{b_j\}_{j\in \widetilde{J}} = \bigcup_{(\lambda,\mu)} \{b_j^{(\lambda,\mu)}\}_{j\in J'}$$

$$\{d_k\}_{k\in \widetilde{K}} = \bigcup_{(\lambda,\mu)} \{d_k^{(\lambda,\mu)}\}_{k\in K'}$$

Theorem. *For any consistent choice as above, there is a rational function* $A : \mathbb{C} \to GL(V)$ *such that*

- $A = 1 + A_0 u^{-1} + \cdots$
- $[A(u), A(v)] = 0$
- *the connection matrix of the system defined by* $A(u)$ *is* $S(z)$
- *the poles of* $A(u)$ *(resp.* $A(u)^{-1}$*) are* $\{b_j\}\cup\{d_k\}$ *(resp.* $\{a_i\}\cup\{d_k\}$*).*

Moreover, if none of the sets $\{a_i\}, \{b_j\}, \{d_k\}$ *are congruent to each other, or to themselves,* A *is unique.*

PROOF. The uniqueness of $A(u)$ follows from Proposition 4.11. To prove the existence of $A(u)$, let $A_0 = A_0^S + A_0^N$ be the Jordan decomposition of A_0. Since

$$S_S(\infty) = e^{\pi \iota A_0^S} = S_S(0)^{-1} \qquad \text{and} \qquad S_U(\infty) = e^{\pi \iota A_0^N} = S_U(0)^{-1}$$

it suffices to treat the cases when $S(z)$ is semisimple and unipotent separately, which we do in 4.15 and 4.16 respectively. $\square$

4.15. **Semisimple case.** Assume that A_0 and $S(z)$ are semisimple, let $\mathfrak{a} \subset \text{End}(V)$ be the abelian Lie algebra spanned by A_0 and $S(w)$, $w \in \mathbb{P}^1$, and decompose $V = \bigoplus_{\lambda \in \mathfrak{a}^*} V_\lambda$ as the direct sum of weight spaces under $\mathfrak{a}$. Since it suffices to factorise $S(z)$ on each V_λ, the problem reduces to the scalar case. We may therefore assume that

$$A_0 \in \mathbb{C} \qquad \text{and} \qquad S(z) = e^{\pi \iota A_0} \prod_i \frac{z - \alpha_i}{z - \beta_i}$$

where α_i, β_i are such that $\prod_i \alpha_i \beta_i^{-1} = e^{-2\pi \iota A_0}$, so that $S(0) = e^{-\pi \iota A_0}$. Let $a_i, b_i \in \mathbb{C}$ be such that

$$e^{2\pi \iota a_i} = \alpha_i \qquad e^{2\pi \iota b_i} = \beta_i \qquad \sum_i b_i - a_i = A_0$$

Then, by Example 4.9, the factorization problem is solved by

$$\phi^+ = \prod_i \frac{\Gamma(u - a_i)}{\Gamma(u - b_i)} \qquad \text{and} \qquad \phi^- = \prod_i \frac{\Gamma(1 - u + b_i)}{\Gamma(1 - u + a_i)}$$

The corresponding coefficient matrix is

$$A(u) = \phi^+(u + 1)\phi^+(u)^{-1} = \prod_i \frac{u - a_i}{u - b_i}$$

4.16. **Unipotent case.** If A_0 is nilpotent, and $S(z)$ unipotent, taking logarithms reduces the factorization problem to finding meromorphic functions $\varphi^\pm : \mathbb{C} \to \text{End}(V)$ such that

(a) $\varphi^\pm$ is holomorphic for $\pm \text{Re}(u) \gg 0$
(b) $\varphi^\pm(u)$ has an asymptotic expansion of the form

$$\varphi^\pm(u) \sim A_0 \log(\pm u) + h_0^\pm u^{-1} + h_1^\pm u^{-2} + \cdots$$

valid in any right (resp. left) half–plane.
(c) $\varphi^-(u) - \varphi^+(u) = \log(S(e^{2\pi \iota u}))$

Since $S(\infty) = e^{\pi \iota A_0} = S(0)^{-1}$, it follows that

$$\log(S(z)) = \pi \iota A_0 + \sum_k \sum_{r \geq 1} \frac{C_{r,k}}{(z - \delta_k)^r}$$

where $\{\delta_k\}$ are the poles of $S(z)$, and $C_{r,k} \in \text{End}(V)$ are such that

$$\sum C_{r,k}(-\delta_k)^{-r} = -2\pi \iota A_0 \tag{4.11}$$

To obtain an additive factorization of $\log(S(z))$, we first determine mero-morphic functions $\varphi_r^{\pm}(u, d)$ for any $r \geq 1$ and $d \in \mathbb{C}$, which are holomorphic for $\pm \operatorname{Re}(u) \gg 0$, and such that

$$\varphi_r^-(u, d) - \varphi_r^+(u, d) = \frac{1}{(z - \delta)^r} \tag{4.12}$$

where $z = e^{2\pi \iota u}$ and $\delta = e^{2\pi \iota d}$.

Set $\Psi^+(u) = \Gamma'(u)/\Gamma(u)$ and $\Psi^-(u) = \Gamma'(1 - u)/\Gamma(1 - u)$. Taking loga-rithmic derivatives in $\Gamma(1 - u)\Gamma(u) = \pi/\sin(\pi u)$, we get

$$\Psi^-(u) - \Psi^+(u) = \pi \cot(\pi u) = \pi \iota \left(\frac{2}{z - 1} + 1 \right)$$

so that the functions

$$\varphi_1^{\pm}(u, d) = \frac{1}{2\delta} \left(\frac{\Psi^{\pm}(u - d)}{\pi \iota} \pm \frac{1}{2} \right)$$

solve (4.12) for $r = 1$. Next, if $\varphi_r^{\pm}(u, d)$ solve (4.12) for a given r, then

$$\partial_u \varphi_r^-(u, d) - \partial_u \varphi_r^+(u, d) = -\frac{2\pi \iota r z}{(z - \delta)^{r+1}}$$

$$= -2\pi \iota r \left(\frac{\delta}{(z - \delta)^{r+1}} + \frac{1}{(z - \delta)^r} \right)$$

so that $\varphi_{r+1}^{\pm}(u, d) = -\dfrac{1}{\delta} \left(\dfrac{\partial_u}{2\pi \iota r} + 1 \right) \varphi_r^{\pm}(u, d)$ solve (4.12) for $r + 1$. We therefore recursively get

$$\varphi_r^{\pm}(u, d) = -\frac{(-\delta)^{-r}}{2} \prod_{k=1}^{r-1} \left(\frac{\partial_u}{2\pi \iota k} + 1 \right) \left(\frac{\Psi^{\pm}(u - d)}{\pi \iota} \pm \frac{1}{2} \right) \tag{4.13}$$

Since $\Psi^{\pm}(u) = \log(\pm u) + O(u^{-1})$ in any right (resp. left) half–plane [53, §12.33], where $O(u^{-1})$ denotes an element of $u^{-1}\mathbb{C}[[u^{-1}]]$, it follows that

$$\varphi_r^{\pm}(u, d) \sim -(-\delta)^{-r} \left(\frac{\log(\pm u)}{2\pi \iota} \pm \frac{1}{4} \right) + O(u^{-1})$$

Choose now $d_k \in \mathbb{C}$ such that $e^{2\pi \iota d_k} = \delta_k$, and set

$$\varphi^{\pm}(u) = \mp \frac{\pi \iota A_0}{2} + \sum_{k,r} \varphi_r^{\pm}(u, d_k) C_{k,r}$$

then $\varphi^-(u) - \varphi^+(u) = \log S(z)$ and, in any right (resp. left) half–plane,

$$\varphi^{\pm} \sim \mp \frac{\pi \iota A_0}{2} - \left(\frac{\log(\pm u)}{2\pi \iota} \pm \frac{1}{4} \right) \sum C_{k,r} (-\delta_k)^{-r} + O(u^{-1})$$

$$= \mp \frac{\pi \iota A_0}{2} + 2\pi \iota A_0 \left(\frac{\log(\pm u)}{2\pi \iota} \pm \frac{1}{4} \right) + O(u^{-1})$$

$$= A_0 \log(\pm u) + O(u^{-1})$$

where we used (4.11), as required.

To determine the coefficient matrix of the difference equation, let Δ be the operator defined by $\Delta f(u) = f(u+1) - f(u)$. Since $\Delta\Psi(u) = 1/u$, (4.13) yields

$$\Delta\varphi_r^{\pm}(u,d) = -\frac{(-\delta)^{-r}}{2\pi\iota}\prod_{k=1}^{r-1}\left(\frac{\partial_u}{2\pi\iota k} + 1\right)\frac{1}{u-d}$$

so that $A(u) = \exp(\Delta\varphi^{\pm})$ is given by

$$A(u) = \exp\left(-\frac{1}{2\pi\iota}\sum_{k,r}(-\delta_k)^{-r}\prod_{k=1}^{r-1}\left(\frac{\partial_u}{2\pi\iota k} + 1\right)\frac{C_{k,r}}{u-d_k}\right)$$

5. The functor Γ

In this section, we construct a functor Γ from a dense subcategory of $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$ to $\mathcal{O}_{\text{int}}(U_q(L\mathfrak{g}))$.

5.1. Non–congruent representations.
We shall say that $V \in \mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$ is *non–congruent* if, for any $i \in \mathbf{I}$, the poles of $x_i^+(u)$ (resp. $x_i^-(u)$) are not congruent modulo $\mathbb{Z}$. Let $\mathcal{O}_{\text{int}}^{\text{NC}}(Y_\hbar(\mathfrak{g}))$ be the full subcategory of $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$ consisting of non–congruent representations.

5.2. Contours.
By a Jordan curve $\mathcal{C}$, we shall mean a disjoint union of simple, closed curves in $\mathbb{C}$, the inner domains of which are pairwise disjoint. By definition, the inner (resp. outer) domain of $\mathcal{C}$ is the union (resp. intersection) of those of its connected components. If $\mathcal{C}$ is a Jordan curve, and f a continuous function on $\mathcal{C}$, we shall set

$$\oint_{\mathcal{C}} f(u)du = \frac{1}{2\pi\iota}\int_{\mathcal{C}} f(u)du$$

The definition of Γ relies upon the following contours of integration. Given $V \in \mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$, a weight $\mu \in \mathfrak{h}^*$ of V, and $i \in \mathbf{I}$, we shall denote by $\mathcal{C}_{i,\mu}^{\pm} \subset \mathbb{C}$ a Jordan curve such that

(i) $\mathcal{C}_{i,\mu}^{\pm}$ encloses the poles of $x_i^{\pm}(u)_\mu$.

(ii) $\mathcal{C}_{i,\mu}^{\pm}$ encloses no $\mathbb{Z}^{\times}$–translate of the poles of $x_i^{\pm}(u)_\mu$ and $x_i^{\pm}(u)_{\mu\pm\alpha_i}$.

Note that such a curve exists for any $i \in \mathbf{I}$ and weight $\mu \in \mathfrak{h}^*$ if V is non–congruent.

5.3. Quantum loop operators.
Given $V \in \mathcal{O}_{\text{int}}^{\text{NC}}(Y_\hbar(\mathfrak{g}))$, and a weight μ of V, we define the action of the generators of $U_q(L\mathfrak{g})$ on the μ–weight space V_μ of $\Gamma(V) = V$ as follows.

(i) For any $h \in \mathfrak{h}$, set $(K_h)_\mu = q^{\mu(h)}$.

(ii) For any $i \in \mathbf{I}$, consider the abelian, additive difference equation

$$\phi_i(u+1)_\mu = \xi_i(u)_\mu \phi_i(u)_\mu \tag{5.1}$$

with coefficient matrix $\xi_i(u)_\mu = 1 + \hbar\xi_{i,0}u^{-1} + \cdots$. Let $S_i(u)_\mu$ be its connection matrix, as defined in 4.8. Set

$$\Psi_i(z)_\mu = S_i(u)_\mu|_{e^{2\pi\iota u} = z}$$

and let $\Psi_i(z)_\mu^\pm \in \mathrm{End}(V_\mu)[[z^{\mp 1}]]$ be its Taylor expansions at ∞ and 0 respectively.

To define the action of the remaining generators of $U_q(L\mathfrak{g})$, consider the fundamental solutions $\phi_i^\pm(u)_\mu$ of (5.1), as defined in 4.2. Define $g_i^\pm(u)_\mu :$ $\mathbb{C} \to GL(V_\mu)$ by

$$g_i^-(u)_\mu = \phi_i^-(u)_\mu \qquad \text{and} \qquad g_i^+(u)_\mu = \phi_i^+(u+1)_\mu^{-1} \tag{5.2}$$

Note that, by (4.7)–(4.8),

$$g_i^-(u)_\mu = e^{-\gamma\hbar\xi_{i,0}} \overrightarrow{\prod_{n\geq 1}} \xi_i(u-n)_\mu \, e^{\hbar\xi_{i,0}/n} \tag{5.3}$$

$$g_i^+(u)_\mu = \overleftarrow{\prod_{n\geq 1}} e^{-\hbar\xi_{i,0}/n} \xi_i(u+n)_\mu \, e^{\gamma\hbar\xi_{i,0}} \tag{5.4}$$

(iii) Let $c_i^\pm \in \mathbb{C}^\times$ be scalars such that

$$c_i^- c_i^+ = d_i \Gamma(\hbar d_i)^2 \tag{5.5}$$

For any $i \in \mathbf{I}$ and $k \in \mathbb{Z}$, the action of $\mathcal{X}_{i,k}^\pm$ as an operator $V_\mu \to V_{\mu\pm\alpha_i}$ is given by

$$(\mathcal{X}_{i,k}^\pm)_\mu = c_i^\pm \oint_{\mathcal{C}_{i,\mu}^\pm} e^{2\pi\iota k u} g_i^\pm(u)_{\mu\pm\alpha_i} x_i^\pm(u)_\mu \, du \tag{5.6}$$

where the contours $\mathcal{C}_{i,\mu}^\pm$ are chosen as in 5.2.

Remark. The above definition implies that the generating functions

$$\mathcal{X}_i^\pm(z)_\mu^+ = \sum_{k\geq 0} \mathcal{X}_{i,k}^\pm z^{-k} \qquad \text{and} \qquad \mathcal{X}_i^\pm(z)_\mu^- = -\sum_{k<0} \mathcal{X}_{i,k}^\pm z^{-k}$$

are the expansions at $z = \infty$ and $z = 0$ of the rational function

$$\mathcal{X}_i^\pm(z)_\mu = c_i^\pm \oint_{\mathcal{C}_{i,\mu}^\pm} \left(\frac{z}{z - e^{2\pi\iota u}} \right) g_i^\pm(u)_{\mu\pm\alpha_i} x_i^\pm(u)_\mu \, du \tag{5.7}$$

where z lies outside the contour $\exp(2\pi\iota\mathcal{C}_{i,\mu}^\pm)$.

Remark. The operator (5.6) is independent of the choice of the contour $\mathcal{C}_{i,\mu}^\pm$ satisfying the conditions of §5.2. Indeed, by (5.3)–(5.4) and Lemma 3.7, the poles of $g_i^\pm(u)_{\mu\pm\alpha_i}$ are contained in the $\mathbb{Z}_{\leqslant 0}$–translates of the poles of $x_i^\pm(u)_\mu$ and $x_i^\pm(u)_{\mu\pm\alpha_i}$. In particular, $g_i^\pm(u)$ is holomorphic inside $\mathcal{C}_{i,\mu}^\pm$.

It also follows that, for any finite collection of weights $\Xi \subset \mathfrak{h}^*$ of V, a contour $\mathcal{C}_i^{\pm}$ can be chosen which satisfies (i)–(ii) of §5.2 for all $\mu \in \Xi$.

5.4.

Theorem.

(i) *The above operators give rise to an action of $U_q(L\mathfrak{g})$ on V. They therefore define an exact, faithful functor*

$$\Gamma : \mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g})) \longrightarrow \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$$

(ii) *The functor Γ is compatible with shift automorphisms. That is, for any $V \in \mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g}))$ and $a \in \mathbb{C}$,*

$$\Gamma(V(a)) = \Gamma(V)(e^{2\pi \iota a})$$

(iii) *Let $\Pi \subset \mathbb{C}$ be a non–congruent subset such that $\Pi \pm \frac{1}{2}\hbar \subset \Pi$. Then, $\mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$ is a subcategory of $\mathcal{O}_{\mathrm{int}}^{\mathrm{NC}}(Y_\hbar(\mathfrak{g}))$, and Γ restricts to a functor*

$$\Gamma_\Pi : \mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g})) \longrightarrow \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$$

where $\Omega = \exp(2\pi \iota \Pi)$.

PROOF. (i) The relations (QL0), (QL1) and (QL2) clearly hold. We prove (QL3) in 5.6, (QL4) in 5.7 and (QL5) in 5.9. Relation (QL6) is a consequence of Proposition 2.13.

A word on notation: the relations of $U_q(L\mathfrak{g})$ need to be verified on a given weight space. For notational convenience, we omit the subscript indicating the weight space in the proofs below. Moreover, the contours $\mathcal{C}_i^{\pm}$ are chosen to satisfy the conditions of §5.2 for all the weight spaces under consideration, consistently with Remark 5.3.

(ii) is proved in 5.10.

(iii) follows from Theorem 3.8 and the monodromy computation in 4.9. $\qquad\square$

5.5.   Set $\Gamma^{\pm}(u) = \Gamma(1 \pm u)$, and note that the identity (4.9) reads

$$\Gamma^-(u)\Gamma^+(u) = \frac{2\pi \iota u}{e^{\pi \iota u} - e^{-\pi \iota u}} \tag{5.8}$$

Let $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij}/2$. We shall need the following

Proposition. *Let $\mathcal{C}$ be a Jordan curve with interior domain D, $\Omega_1, \Omega_2 \subset \mathbb{C}$ two open subsets with $\overline{D} \subset \Omega_2$, and $f : \Omega_1 \times \Omega_2 \to \mathrm{End}(V)$ a holomorphic function such that $[\xi_i(u), f(u, v)] = 0$ for any u, v. Then, the following holds for any $\epsilon, \eta \in \{\pm\}$.*

(i) *If $u \notin \overline{D} \pm \epsilon a$,*

$$\mathrm{Ad}(\xi_i(u))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\, x_j^\epsilon(v) dv = \oint_{\mathcal{C}} \left(\frac{u - v + \epsilon a}{u - v - \epsilon a} \right)^{\pm 1} f(u,v)\, x_j^\epsilon(v) dv$$

(ii) *If $u \notin \overline{D} \pm \epsilon a - \eta \mathbb{N}^\times$*

$$\mathrm{Ad}(g_i^\eta(u))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\, x_j^\epsilon(v) dv = \oint_{\mathcal{C}} \left(\frac{\Gamma^\eta(u - v - \epsilon a)}{\Gamma^\eta(u - v + \epsilon a)} \right)^{\pm 1} f(u,v)\, x_j^\epsilon(v) dv$$

(iii) *If $u \notin \overline{D} \pm \epsilon a + \mathbb{Z}$*

$$\mathrm{Ad}(S_i(u))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\, x_j^\epsilon(v) dv = \oint_{\mathcal{C}} \left(\frac{q_i^{\epsilon a_{ij}} z - w}{z - q_i^{\epsilon a_{ij}} w} \right)^{\pm 1} f(u,v)\, x_j^\epsilon(v) dv$$

   where $z = e^{2\pi \iota u}$ and $w = e^{2\pi \iota v}$.

PROOF. (i) follows by left multiplying ($\mathcal{Y}3$) by $f(u,v)$ and integrating along $\mathcal{C}$.

   (ii) A repeated application of (i) shows that, for any $n \geq 1$

$$\mathrm{Ad}(\xi_i(u + \eta n) \cdots \xi_i(u + \eta 1))^{\pm 1} \oint_{\mathcal{C}} f(u,v)\, x_j^\epsilon(v) dv$$
$$= \oint_{\mathcal{C}} \prod_{m=1}^n \left(\frac{u + \eta m - v + \epsilon a}{u + \eta m - v - \epsilon a} \right)^{\pm 1} f(u,v)\, x_j^\epsilon(v) dv$$

so long as $u \notin \overline{D} \pm \epsilon a - \{\eta, \dots, \eta n\}$. The result then follows from (5.3)–(5.4), $[\hbar \xi_{i,0}, x_j^\epsilon(v)] = \epsilon 2a x_j^\epsilon(v)$ and the fact that, for any $\alpha, \beta \in \mathbb{C}$

$$e^{\gamma(\alpha - \beta)} \prod_{m=1}^n \frac{m + \alpha}{m + \beta} e^{-\frac{\alpha - \beta}{m}} \longrightarrow \frac{\beta}{\alpha} \frac{\Gamma(\beta)}{\Gamma(\alpha)} = \frac{\Gamma^+(\beta)}{\Gamma^+(\alpha)} = \frac{\Gamma^-(-\beta)}{\Gamma^-(-\alpha)}$$

   (iii) follows from (i)–(ii), the fact that

$$S_i(u) = g_i^+(u) \xi_i(u) g_i^-(u) \tag{5.9}$$

and (5.8). $\square$

5.6. **Proof of (QL3).** We shall check that

$$z \left(\mathrm{Ad}(\Psi_i(z)) \mathcal{X}_{j,\ell}^\pm - q_i^{\pm a_{ij}} \mathcal{X}_{j,\ell}^\pm \right) = q_i^{\pm a_{ij}} \mathrm{Ad}(\Psi_i(z)) \mathcal{X}_{j,\ell+1}^\pm - \mathcal{X}_{j,\ell+1}^\pm \tag{5.10}$$

The relation (QL3) then follows by expanding around $z = \infty$ and $z = 0$.

   Since (5.10) is an equality of rational functions, it suffices to prove it when z lies outside of $\exp(2\pi \iota \mathcal{C}_j^\pm)$. Since $g_j^\pm$ is holomorphic inside $\mathcal{C}_j^\pm$ by Remark 5.3, Proposition 5.5 implies that, for any $k \in \mathbb{Z}$,

$$\mathrm{Ad}(\Psi_i(z)) \mathcal{X}_{j,k}^\pm = c_j^\pm \oint_{\mathcal{C}_j^\pm} w^k \frac{q_i^{\pm a_{ij}} z - w}{z - q_i^{\pm a_{ij}} w} g_j^\pm(v) x_j^\pm(v) dv$$

where $w = \exp(2\pi\iota v)$. The left–hand side of (5.10) is therefore equal to

$$c_j^\pm \oint_{\mathcal{C}_j^\pm} zw^\ell \left(\frac{q_i^{\pm a_{ij}} z - w}{z - q_i^{\pm a_{ij}} w} - q_i^{\pm a_{ij}} \right) g_j^\pm(v) x_j^\pm(v) dv$$

and the right–hand side to

$$c_j^\pm \oint_{\mathcal{C}_j^\pm} w^{\ell+1} \left(q_i^{\pm a_{ij}} \frac{q_i^{\pm a_{ij}} z - w}{z - q_i^{\pm a_{ij}} w} - 1 \right) g_j^\pm(v) x_j^\pm(v) dv$$

5.7. **Proof of (QL4).** We shall prove (QL4) in the form

$$\mathcal{X}_{i,k+1}^\pm \mathcal{X}_{j,\ell}^\pm - q_i^{\pm a_{ij}} \mathcal{X}_{i,k}^\pm \mathcal{X}_{j,\ell+1}^\pm = q_i^{\pm a_{ij}} \mathcal{X}_{j,\ell}^\pm \mathcal{X}_{i,k+1}^\pm - \mathcal{X}_{j,\ell+1}^\pm \mathcal{X}_{i,k}^\pm \qquad (5.11)$$

Set $a = \hbar d_i a_{ij}/2$, so that $\exp(2\pi\iota a) = q_i^{a_{ij}}$. By Proposition 5.5,

$$\oint_{\mathcal{C}_i^\pm} \left(e^{2\pi\iota(k+1)u} e^{2\pi\iota\ell v} - q_i^{\pm a_{ij}} e^{2\pi\iota k u} e^{2\pi\iota(\ell+1)v} \right) g_i^\pm(u) x_i^\pm(u) du \cdot g_j^\pm(v)$$

$$= g_j^\pm(v) \cdot \oint_{\mathcal{C}_i^\pm} e^{2\pi\iota(ku+\ell v)} \left(e^{2\pi\iota u} - e^{2\pi\iota(v\pm a)} \right) \frac{\Gamma^\pm(v - u \pm a)}{\Gamma^\pm(v - u \mp a)} g_i^\pm(u) x_i^\pm(u) du$$

$$(5.12)$$

so long as $v \notin \overline{D}_i^\pm \mp a \mp \mathbb{N}^\times$. As a function of v, the integrand on the right–hand side of (5.12) is regular on $\mathcal{C}_i^\pm \mp a \mp \mathbb{N}^\times$ since the (simple) poles of $\Gamma^\pm(v - u \pm a)$ are cancelled by the zeros of $\left(e^{2\pi\iota u} - e^{2\pi\iota(v\pm a)} \right)$. It follows that (5.12) holds for any v.

Right multiplying the above by $x_j^\pm(v)$ and integrating along $v \in \mathcal{C}_j^\pm$ then shows that the left–hand side of (5.11) is equal to

$$c_i^\pm c_j^\pm \oint_{\mathcal{C}_j^\pm} \oint_{\mathcal{C}_i^\pm} e^{2\pi\iota(ku+\ell v)} \left(e^{2\pi\iota u} - e^{2\pi\iota(v\pm a)} \right)$$

$$\cdot \frac{\Gamma^\pm(v - u \pm a)}{\Gamma^\pm(v - u \mp a)} g_i^\pm(u) g_j^\pm(v) x_i^\pm(u) x_j^\pm(v) dudv$$

Similarly, the right–hand side is equal to

$$c_i^\pm c_j^\pm \oint_{\mathcal{C}_j^\pm} \oint_{\mathcal{C}_i^\pm} e^{2\pi\iota(ku+\ell v)} \left(e^{2\pi\iota(u\pm a)} - e^{2\pi\iota v} \right)$$

$$\cdot \frac{\Gamma^\pm(u - v \pm a)}{\Gamma^\pm(u - v \mp a)} g_i^\pm(u) g_j^\pm(v) x_j^\pm(v) x_i^\pm(u) dudv$$

By ($\mathcal{Y}4$), the latter expression may be rewritten as

$$c_i^\pm c_j^\pm \oint_{\mathcal{C}_j^\pm} \oint_{\mathcal{C}_i^\pm} e^{2\pi\iota(ku+\ell v)} \left(e^{2\pi\iota(u\pm a)} - e^{2\pi\iota v} \right)$$

$$\cdot \frac{v - u \pm a}{v - u \mp a} \frac{\Gamma^\pm(u - v \pm a)}{\Gamma^\pm(u - v \mp a)} g_i^\pm(u) g_j^\pm(v) x_i^\pm(u) x_j^\pm(v) dudv$$

since the contribution of the terms $[x_{i,0}^{\pm}, x_j^{\pm}(v)]$ and $[x_i^{\pm}(u), x_{j,0}^{\pm}(v)]$ to the double integral is zero given that $(e^{2\pi\iota(u\pm a)} - e^{2\pi\iota v})\Gamma^{\pm}(u - v \pm a)$ is holomorphic in $D_i \times D_j$.

(QL4) now follows from the fact that $\Gamma^+(x) = \Gamma^-(-x)$ and (5.8).

5.8. We shall need the following result to show that relation (QL5) holds.

Lemma. *Let $\mathcal{C} \subset \mathbb{C}$ be a Jordan curve with interior domain D, $\xi(u)$ a meromorphic function with no singularities on $\mathcal{C}$, and $f(u, v)$ a meromorphic function with no singularities in $\overline{D} \times \overline{D}$. Then,*

$$\oint_{\mathcal{C}} \oint_{\mathcal{C}} f(u, v) \frac{\xi(u) - \xi(v)}{u - v} du\, dv = -\oint_{\mathcal{C}} f(u, u)\xi(u) du$$

PROOF. Let $\mathcal{C}_<$ be a small deformation of $\mathcal{C}$ contained in $\mathcal{C}$. Then,

$$
\begin{aligned}
\oint_{\mathcal{C}} \oint_{\mathcal{C}} f(u, v) \frac{\xi(u) - \xi(v)}{u - v} du\, dv &= \oint_{\mathcal{C}} \left(\oint_{\mathcal{C}_<} f(u, v) \frac{\xi(u) - \xi(v)}{u - v} du \right) dv \\
&= \oint_{\mathcal{C}} \left(\oint_{\mathcal{C}_<} f(u, v) \frac{\xi(u)}{u - v} du \right) dv \\
&= \oint_{\mathcal{C}_<} \xi(u) \left(\oint_{\mathcal{C}} \frac{f(u, v)}{u - v} dv \right) du \\
&= -\oint_{\mathcal{C}_<} \xi(u) f(u, u) du
\end{aligned}
$$

$\square$

5.9. **Proof of (QL5).** We have

$$
\begin{aligned}
[\mathcal{X}_{i,k}^+, \mathcal{X}_{j,\ell}^-] &= c_i^+ c_j^- \oint_{\mathcal{C}_i^+} \oint_{\mathcal{C}_j^-} e^{2\pi\iota ku} e^{2\pi\iota \ell v} g_i^+(u) x_i^+(u) g_j^-(v) x_j^-(v) du dv \\
&\quad - c_i^+ c_j^- \oint_{\mathcal{C}_j^-} \oint_{\mathcal{C}_i^+} e^{2\pi\iota ku} e^{2\pi\iota \ell v} g_j^-(v) x_j^-(v) g_i^+(u) x_i^+(u) du dv
\end{aligned}
$$

We wish to apply Proposition 5.5 to permute $x_i^+(u)$ and $g_j^-(v)$ in the first integral, and $x_j^-(v)$ and $g_i^+(u)$ in the second. This cannot be done directly however, since $\mathcal{C}_j^- \ni v$ may lie inside $\overline{D}_i^+ - a + \mathbb{N}^\times$, where $a = \hbar d_i a_{ij}/2$, and $\mathcal{C}_i^+ \ni u$ may lie inside $\overline{D}_j^- + a - \mathbb{N}^\times$.

To circumvent this issue, let $\delta \in \mathbb{C}$ and consider the shifted integral

$$
\begin{aligned}
I(\delta) &= c_i^+ c_j^- \oint_{\mathcal{C}_i^+} \oint_{\mathcal{C}_j^-} e^{2\pi\iota ku} e^{2\pi\iota \ell v} g_i^+(u + \delta) x_i^+(u) g_j^-(v - \delta) x_j^-(v) du dv \\
&\quad - c_i^+ c_j^- \oint_{\mathcal{C}_j^-} \oint_{\mathcal{C}_i^+} e^{2\pi\iota ku} e^{2\pi\iota \ell v} g_j^-(v - \delta) x_j^-(v) g_i^+(u + \delta) x_i^+(u) dv du
\end{aligned}
$$

$I(\delta)$ is holomorphic in a disk $|\delta| < R$, where R is such that g_i^+ and g_j^- are holomorphic on $\overline{D}_i^+ + \delta'$ and $\overline{D}_j^- - \delta'$ for any δ' such that $|\delta'| < R$ respectively.

Moreover, if $\mathcal{C}_i^+, \mathcal{C}_j^-$ are small enough, there is an $r < R$ such that if $|\delta| > r$, $\overline{D}_j^- - \delta$ is disjoint from $\overline{D}_i^+ - a + \mathbb{N}^\times$, and $\overline{D}_i^+ + \delta$ is disjoint from $\overline{D}_j^- + a - \mathbb{N}^\times$.

Assuming that $r < |\delta| < R$, we may now apply Proposition 5.5 to find that $I(\delta)$ is equal to

$$c_i^+ c_j^- \oint_{\mathcal{C}_i^+} \oint_{\mathcal{C}_j^-} e^{2\pi \iota k u} e^{2\pi \iota \ell v} g_i^+(u+\delta) g_j^-(v-\delta) \frac{\Gamma^-(v - \delta - u + a)}{\Gamma^-(v - \delta - u - a)} x_i^+(u) x_j^-(v) du dv$$

$$-c_i^+ c_j^- \oint_{\mathcal{C}_j^-} \oint_{\mathcal{C}_i^+} e^{2\pi \iota k u} e^{2\pi \iota \ell v} g_j^-(v-\delta) g_i^+(u+\delta) \frac{\Gamma^+(u + \delta - v - a)}{\Gamma^+(u + \delta - v + a)} x_j^-(v) x_i^+(u) dv du$$

Since $\Gamma^+(x) = \Gamma^-(x)$, this yields

$$\frac{I(\delta)}{c_i^+ c_j^-} = \oint_{\mathcal{C}_i^+} \oint_{\mathcal{C}_j^-} e^{2\pi \iota k u} e^{2\pi \iota \ell v} g_i^+(u + \delta) g_j^-(v - \delta) \frac{\Gamma^+(u + \delta - v - a)}{\Gamma^+(u + \delta - v + a)} [x_i^+(u), x_j^-(v)] du dv$$

$$= -\hbar \delta_{ij} \oint_{\mathcal{C}_i^+} \oint_{\mathcal{C}_i^-} e^{2\pi \iota k u} e^{2\pi \iota \ell v} g_i^+(u + \delta) g_i^-(v - \delta) \frac{\Gamma^+(u + \delta - v - \hbar d_i)}{\Gamma^+(u + \delta - v + \hbar d_i)} \frac{\xi_i(u) - \xi_i(v)}{u - v} du dv$$

where we used $(\mathcal{Y}5)$.

Since the function $\Gamma^+(u + \delta - v - \hbar d_i)$ is holomorphic on $\overline{D}_i^+ \times \overline{D}_i^-$ for $|\delta| > r$, the only singularities of the integrand are the poles of $\xi_i(u), \xi_i(v)$. Choosing the contours $\mathcal{C}_i^\pm$ to be equal to a common contour $\mathcal{C}_i$ which contains all poles of ξ_i, Lemma 5.8 then yields

$$I(\delta) = c_i^+ c_i^- \hbar \delta_{ij} \oint_{\mathcal{C}_i} e^{2\pi \iota (k+\ell) u} g_i^+(u + \delta) g_i^-(u - \delta) \frac{\Gamma^+(\delta - \hbar d_i)}{\Gamma^+(\delta + \hbar d_i)} \xi_i(u) du \quad (5.13)$$

Since both $I(\delta)$ and the right–hand side $J(\delta)$ of (5.13) are holomorphic functions on the disk $|\delta| < R$, $I(0)$ is equal to

$$J(0) = \delta_{ij} \hbar c_i^+ c_i^- \frac{\Gamma(1 - d_i \hbar)}{\Gamma(1 + d_i \hbar)} \oint_{\mathcal{C}_i} e^{2\pi \iota (k+\ell) u} g_i^+(u) g_j^-(u) \xi_i(u) du$$

$$= \delta_{ij} \frac{2\pi \iota}{q_i - q_i^{-1}} \oint_{\mathcal{C}_i} e^{2\pi \iota (k+\ell) u} S_i(u) du$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}} \oint_{\widetilde{\mathcal{C}}_i} z^{k+\ell} \Psi_i(z) dz$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(\Psi_{i,k+\ell}^+ - \Psi_{i,k+\ell}^- \right)$$

where the second equality follows from (5.9), (5.5) and (5.8), the third from the change of variables $z = \exp(2\pi \iota u)$ [7], and the fourth from the fact that $\widetilde{\mathcal{C}}_i = \exp(2\pi \iota \mathcal{C}_i)$ encloses all the poles of Ψ_i by 5.2 (i), and neither $z = 0$ nor $z = \infty$.

---

[7] we are assuming $\mathcal{C}_i$ to be small enough that the exponential map is injective on each of the connected components of its interior domain D_i

5.10. Compatibility with shift automorphisms.

Let $\rho : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V)$ be the action of $Y_\hbar(\mathfrak{g})$ on V and $\rho_a : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V)$ the pull–back of ρ via the shift automorphism τ_a of $Y_\hbar(\mathfrak{g})$. Thus, for $y = \xi_i, x_i^\pm$ we have

$$\rho_a(y(u)) = \rho(y(u-a))$$

We need to show that for $Y = \Psi_i, \mathcal{X}_i^\pm$, the following holds on V

$$\Gamma(\rho_a)(Y(z)) = \Gamma(\rho)(Y(e^{-2\pi \iota a} z))$$

For notational convenience, we denote the left–hand side by $\widetilde{Y}(z)$ and the right–hand side by $Y(\alpha^{-1} z)$. By definition, $\widetilde{\Psi}_i(z)$ is obtained from the monodromy of the difference equation

$$\widetilde{\phi}_i(u+1) = \xi_i(u-a)\widetilde{\phi}_i(u)$$

Clearly if $\phi_i^\pm(u)$ and $S_i(u)$ are the fundamental solutions and the connection matrix of $\phi_i(u+1) = \xi_i(u)\phi_i(u)$, then $\phi_i^\pm(u-a), S_i(u-a)$ are those of the difference equation $\widetilde{\phi}_i(u+1) = \xi_i(u-a)\widetilde{\phi}_i(u)$. Therefore, we get

$$\widetilde{\Psi}_i(z) = S_i(u-a)|_{z=e^{2\pi \iota u}} = S_i(u)|_{z=e^{2\pi \iota (u+a)}} = \Psi_i(e^{-2\pi \iota a} z)$$

Next, if $\mathcal{C}$ is the contour used to define $\mathcal{X}_i^\pm(z)$, then $\mathcal{C} + a$ can be taken as the contour for $\widetilde{\mathcal{X}_i^\pm}(z)$. It follows that

$$\begin{aligned}
\widetilde{\mathcal{X}_i^\pm}(z) &= c_i^\pm \oint_{\mathcal{C}+a} \frac{z}{z - e^{2\pi \iota u}} g_i^\pm(u-a) x_i^\pm(u-a) du \\
&= c_i^\pm \oint_{\mathcal{C}} \frac{z}{z - e^{2\pi \iota(u+a)}} g_i^\pm(u) x_i^\pm(u) du \\
&= \mathcal{X}_i^\pm(e^{-2\pi \iota a} z)
\end{aligned}$$

6. The inverse functor $\daleth$

Let $\Omega \subset \mathbb{C}^\times$ be a subset stable under multiplication by q, and $\Pi \subset \mathbb{C}$ a non–congruent subset stable under shifts by $\frac{1}{2}\mathbb{Z}\hbar$ such that $\Omega = \exp(2\pi \iota \Pi)$. In this section, we construct a functor

$$\daleth_\Pi : \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g})) \to \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$$

which is an inverse to the functor Γ_Π given by Theorem 5.4.

6.1. Inverse monodromy problem.

Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$, and let $\mu \in \mathfrak{h}^*$ be a weight of $\mathcal{V}$. For any $i \in \mathbf{I}$, Proposition 3.6 yields a $GL(\mathcal{V}_\mu)$–valued rational function $\Psi_i(z)_\mu$ satisfying

$$\Psi_i(\infty)_\mu = q_i^{\mu(\alpha_i^\vee)} = \Psi_i(0)_\mu^{-1}$$

and such that the poles of $\Psi_i(z)_\mu^{\pm 1}$ are contained in Π. Set $A_0 = \hbar d_i \mu(\alpha_i^\vee)$ so that $\Psi_i(\infty)_\mu = \exp(\pi \iota A_0)$, and $S_i(u) = \Psi\left(e^{2\pi \iota u}\right)_\mu$.

Lemma. *There exists a unique rational function $A(u) : \mathbb{C} \to GL(\mathcal{V}_\mu)$ such that*

- $A(u) = 1 + A_0 u^{-1} + \cdots$.
- $[A(u), A(v)] = 0$ *for any* u, v.
- *the poles of* $A(u)^{\pm 1}$ *are contained in* Π.
- $S_i(u)$ *is the connection matrix of* $\phi(u+1) = A(u)\phi(u)$.

PROOF. By Theorem 4.14, the inverse monodromy problem may be solved by making a choice of logarithms of the zeros and poles of the eigenvalues of $\Psi_i(z)_\mu$, and of the poles of the unipotent part of $\Psi_i(z)_\mu$. Choosing the logarithms to lie in Π leads to a unique coefficient matrix $A(u)$ since Π is non–congruent. We do however need to check that the choices are consistent in the sense of equation (4.10).

Let $\gamma(z)$ be an eigenvalue of $\Psi_i(z)$ on the weight space $\mathcal{V}_\mu$. By [24],[32, Prop. 16], there are two monic polynomials $Q(z), R(z)$, not vanishing at $z = 0$, and such that

$$\gamma(z) = q_i^{\deg(R)-\deg(Q)} \frac{Q(q_i^2 z)}{Q(z)} \frac{R(z)}{R(q_i^2 z)}$$

Let $Q(z) = \prod_j (z - \beta_j)$ and $R(z) = \prod_k (z - \beta_k')$. Let $b_j, b_k' \in \Pi$ be such that $e^{2\pi\iota b_j} = \beta_j$ and $e^{2\pi\iota b_k'} = \beta_k'$. Since Π is stable under shifts by $\hbar/2$, $b_j - d_i\hbar, b_k' - d_i\hbar$ are logarithms of $q_i^{-2}\beta_j$ and $q_i^{-2}\beta_k'$ respectively, which lie in Π. Since $A_0 = \hbar d_i \mu(\alpha_i^\vee)$, the consistency requirement (4.10) reads

$$\sum_j b_j + \sum_k (b_k' - d_i\hbar) - \sum_j (b_j - d_i\hbar) - \sum_k b_k' = \hbar d_i \mu(\alpha_i^\vee)$$

The latter holds since $\Psi_i(\infty)$ acts on $\mathcal{V}_\mu$ as multiplication by $q_i^{\mu(\alpha_i^\vee)}$, $\gamma(\infty) = q_i^{\deg Q - \deg R}$, and q is not a root of unity. $\qquad\square$

6.2. **Yangian operators.** Given $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$, and a weight $\mu \in \mathfrak{h}^*$ of $\mathcal{V}$, define the action of the generators of $Y_\hbar(\mathfrak{g})$ on $\mathcal{V}_\mu$ as follows.

(i) The action of $\mathfrak{h}$ on $\mathcal{V}_\mu$ is by weight μ. Note that $K_h = q^{\mu(h)}$ on $\mathcal{V}_\mu$ for every $h \in \mathfrak{h}$ determines $\mu \in \mathfrak{h}^*$ uniquely.

(ii) $\xi_i(u)_\mu$ acts as the coefficient matrix $A(u)$ given by Lemma 6.1.

Let $\phi_i^\pm(u)_\mu$ be the fundamental solutions of the difference equation determined by $\xi_i(u)_\mu$, and define $g_i^\pm(u)_\mu$ by (5.2). Let $\mathcal{C}_{i,\mu} \subset \mathbb{C}$ a Jordan curve such that

(a) $\mathcal{C}_{i,\mu}^\pm$ encloses all the poles of $\mathcal{X}_i^\pm(e^{2\pi\iota u})$ contained in Π, and none of their $\mathbb{Z}^\times$–translates.

(b) $\mathcal{C}_{i,\mu}^\pm$ encloses none of the $\mathbb{Z}^\times$–translates of the poles of $\xi_i(u)_\mu^{-1}$.

Note that such a curve exists since the poles of $\xi_i(u)_\mu^{-1}$ are contained in Π, and the latter is non–congruent.

(iii) The generators $x_{i,r}^\pm$ act on $\mathcal{V}_\mu$ by

$$\left(x_{i,r}^\pm\right)_\mu = \frac{1}{c_i^\pm \hbar} \int_{\mathcal{C}_{i,\mu}^\pm} v^r g_i^\pm(v)_{\mu\pm\alpha_i}^{-1} \mathcal{X}_i^\pm \left(e^{2\pi\iota v}\right)_\mu \, dv \qquad (6.1)$$

Note that these operators are independent of the choice of the contour $\mathcal{C}_{i,\mu}$ since, by assumption (b) and (5.3)–(5.4), $g_i^\pm(v)_{\mu\pm\alpha_i}^{-1}$ are holomorphic inside $\mathcal{C}_{i,\mu}$. In terms of the fields $x_i^\pm(u)$, (6.1) may be rewritten as

$$x_i^\pm(u)_\mu = \frac{1}{c_i^\pm} \int_{\mathcal{C}_{i,\mu}^\pm} \frac{1}{u-v} g_i^\pm(v)_{\mu\pm\alpha_i}^{-1} \mathcal{X}_i^\pm \left(e^{2\pi\iota v}\right)_\mu dv \qquad (6.2)$$

for u not enclosed by $\mathcal{C}_{i,\mu}$.

6.3.

Theorem.

  (i) *The above operators define an action of $Y_\hbar(\mathfrak{g})$ on $\mathcal{V}$.*
  (ii) *As a representation of $Y_\hbar(\mathfrak{g})$, $\mathcal{V}$ lies in $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$.*
  (iii) *The corresponding functor $\daleth_\Pi : \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g})) \to \mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$ satisfies*

  $$\Gamma_\Pi \circ \daleth_\Pi = \mathbf{1}_{\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))} \qquad and \qquad \daleth_\Pi \circ \Gamma_\Pi = \mathbf{1}_{\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))}$$

  In particular, if Π is a fundamental domain for $u \mapsto u+1$, Γ_Π gives rise to an isomorphism of categories $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g})) \xrightarrow{\sim} \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$.
  (iv) *The functor $\daleth_\Pi$ is compatible with shift automorphisms. That is, for any $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))$, and $a \in \mathbb{C}$*

  $$\daleth_{\Pi+a}(\mathcal{V}(e^{2\pi\iota a})) = \daleth_\Pi(\mathcal{V})(a)$$

PROOF. (i) The relations (Y0), (Y1) and (Y2) hold trivially. After some preparatory work in 6.5–6.6, we prove (Y3) in §6.7, (Y4) in §6.8 and (Y5) in §6.9. Relation (Y6) follows from Proposition 2.7.

(ii) By construction, $\{\xi_i(u)\}_{i\in\mathbf{I}}$ have zeros and poles in Π. It follows by Theorem 3.8 part (ii) that $\daleth_\Pi(\mathcal{V})$ lies in $\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))$.

(iii) The fact that $\Gamma_\Pi \circ \daleth_\Pi(\mathcal{V}) = \mathcal{V}$ as modules over the subalgebra of $U_q(L\mathfrak{g})$ generated by $\{\Psi_{i,\pm r}^\pm\}_{i\in\mathbf{I},r\in\mathbb{N}}$ follows from the definition of Γ, and the fact that Lemma 6.1 solves an inverse monodromy problem. The fact that $\daleth_\Pi \circ \Gamma_\Pi(V) = V$ as modules over the commutative subalgebra of $Y_\hbar(\mathfrak{g})$ generated by $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ follows from the uniqueness statement of Lemma 6.1. The corresponding assertions for the generators $x_{i,r}^\pm, \mathcal{X}_{i,k}^\pm$ of $Y_\hbar(\mathfrak{g}), U_q(L\mathfrak{g})$ are checked in 6.10 and 6.11 respectively.

(iv) Follows by a direct check similar to that in 5.10, or by applying $\Gamma_{\Pi+a}$ and using Theorem 5.4 (ii). □

6.4.

Remark. We observe that one can always find $\Pi \subset \mathbb{C}$ which is stable under shifts by $\frac{\hbar}{2}$ and is a fundamental domain for $u \mapsto u+1$, assuming that $\hbar$ is not rational. If $\mathrm{Im}(\hbar) \neq 0$, Π may be taken to be the strip bounded between $\mathbb{R}\hbar$ and $\mathbb{R}\hbar + 1$ including $\mathbb{R}\hbar$.

If $\hbar \in \mathbb{R} \setminus \mathbb{Q}$, we claim that there exists $U \subset \mathbb{R}$ such that $U \pm \frac{\hbar}{2} \subset U$ and $u \mapsto e^{2\pi\iota u}$ is a bijection between U and S^1. Assuming this, take $\Pi =$

$\{u + \iota t : u \in U, t \in \mathbb{R}\}$. To prove the claim, we shall use Zorn's lemma as follows. Let $\mathcal{U}$ to be the set of all subsets $V \subset \mathbb{R}$ such that V is stable under shifts by $\hbar/2$ and $u \mapsto e^{2\pi \iota u}$ is an injective map from V to S^1. The set $\mathcal{U}$ is non–empty, since $\mathbb{Z}\frac{\hbar}{2}$ is an element of $\mathcal{U}$. Moreover every chain in $\mathcal{U}$ has the largest element in $\mathcal{U}$ and hence by Zorn's lemma $\mathcal{U}$ has a maximal element U. We claim that $u \mapsto e^{2\pi \iota u}$ is a bijection between U and S^1. If not, then there exists $x \in \mathbb{R}$ such that $x \not\equiv y \pmod{\mathbb{Z}}$ for every $y \in U$. The same is true for every $x + n\hbar$ ($n \in \frac{1}{2}\mathbb{Z}$) and hence $U \cup \{x + n\hbar\}_{n \in (1/2)\mathbb{Z}}$ is also in $\mathcal{U}$, contradicting the maximality of U.

6.5.

Lemma. *Let $\mathcal{C}$ be a Jordan curve the interior domain D of which is non–congruent, and f a meromorphic function on a neighborhood of $\overline{D}$ with no singularities on $\mathcal{C}$. Let $a \in \mathbb{C} \setminus \mathbb{Z}$, and assume that*

$$\int_{\mathcal{C}} \frac{e^{2\pi \iota (u+a)} - e^{2\pi \iota v}}{e^{2\pi \iota u} - e^{2\pi \iota (v+a)}} f(v)\,dv = 0$$

for any $u \notin \overline{D} + a + \mathbb{Z}$. Then, f has no poles in D.

PROOF. Expanding the above integral near $e^{2\pi \iota u} = \infty$ shows that

$$\int_{\mathcal{C}} e^{2\pi \iota k v} f(v)\,dv = 0$$

for any $k \in \mathbb{N}$, and therefore that $\int_{\mathcal{C}} p(e^{2\pi \iota v}) f(v)\,dv = 0$ for any polynomial p. Let $\{b_j\}_{j \in J}$ be the poles of f in D, and $\{n_j\}_{j \in J}$ their orders. For any $j \in J$, set

$$p_j(w) = (w - \beta_j)^{n_j - 1} \prod_{j' \neq j} (w - \beta_{j'})^{n_{j'}}$$

where $\beta_k = e^{2\pi \iota b_k}$. Then $p(e^{2\pi \iota v}) f(v)$ is holomorphic on $D \setminus \{b_j\}$ and has a pole of order 1 at $v = b_j$ since D is non–congruent. It follows that $\int_{\mathcal{C}} p(e^{2\pi \iota v}) f(v)\,dv \neq 0$, a contradiction. $\qquad\square$

6.6.   Let $i, j \in \mathbf{I}$, and set $a = \hbar d_i a_{ij}/2$. The following is a converse to Proposition 5.5.

Proposition. *Let $\mathcal{C}$ be a Jordan curve with interior domain D, and $f : \mathbb{C} \to \mathrm{End}(\mathcal{V})$ a meromorphic function with no singularities in $\overline{D}$, and such that $[\Psi_i(e^{2\pi \iota u}), f(v)] = 0$ for any u, v. Then, the following holds for any $\epsilon \in \{\pm\}$.*

   (i) If $u \notin \overline{D} \pm \epsilon a + \mathbb{Z}$

$$\mathrm{Ad}(\Psi_i(e^{2\pi \iota u}))^{\pm 1} \int_{\mathcal{C}} f(v)\,\mathcal{X}_j^\epsilon(e^{2\pi \iota v})\,dv$$

$$= \int_{\mathcal{C}} \left(\frac{e^{2\pi \iota (u+\epsilon a)} - e^{2\pi \iota v}}{e^{2\pi \iota u} - e^{2\pi \iota (v+\epsilon a)}} \right)^{\pm 1} f(v)\,\mathcal{X}_j^\epsilon(e^{2\pi \iota v})\,dv$$

Assume moreover that D is such that

(1) D is non–congruent.

(2) $\overline{D} + a$ is non–congruent to $\overline{D} - a$.

(3) $\overline{D} \pm a$ are non–congruent to the set of poles of $\xi_i(u)^{\pm 1}$.

Then, the following holds for any $\epsilon, \eta \in \{\pm\}$.

(ii) *If $u \notin \overline{D} \pm \epsilon a - \eta \mathbb{N}^{\times}$*

$$\mathrm{Ad}(g_i^{\eta}(u))^{\pm 1} \int_{\mathcal{C}} f(v)\, \mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv = \int_{\mathcal{C}} \left(\frac{\Gamma^{\eta}(u - v - \epsilon a)}{\Gamma^{\eta}(u - v + \epsilon a)} \right)^{\pm 1} f(v)\, \mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv$$

(iii) *If $u \notin \overline{D} \pm \epsilon a$,*

$$\mathrm{Ad}(\xi_i(u))^{\pm 1} \int_{\mathcal{C}} f(v)\, \mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv = \int_{\mathcal{C}} \left(\frac{u - v + \epsilon a}{u - v - \epsilon a} \right)^{\pm 1} f(v)\, \mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv$$

PROOF. (i) follows by left multiplying $(\mathcal{QL}3)$ by f and integrating along $\mathcal{C}$.

(ii)–(iii) Fix a weight μ of $\mathcal{V}$, and let $E^{\epsilon} \subseteq \mathrm{Hom}(\mathcal{V}_{\mu}, \mathcal{V}_{\mu + \epsilon \alpha_j})$ be the subspace spanned by elements of the form $\int_{\mathcal{C}} f(v)\mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv$, where $f(v)$ is meromorphic with no singularities in $\overline{D}$, and commutes with $\Psi_i(e^{2\pi \iota u})$ for all u, v. By (i), E^{ϵ} is stable under $\mathrm{Ad}(\Psi_i(e^{2\pi \iota u}))$ for $u \notin \overline{D} + \epsilon a + \mathbb{Z}$, and therefore for any u since $\Psi_i(e^{2\pi \iota u})$ is meromorphic in u. It follows that E^{ϵ} is stable under $\mathrm{Ad}(\xi_i(u))$ and $\mathrm{Ad}(g_i^{\eta}(u))$. We now wish to show that the action of $\mathrm{Ad}(g_i^{\eta}(u))$ and $\mathrm{Ad}(\xi_i(u))$ on E^{ϵ} is given by the formulae (ii)–(iii). We shall do so by showing that the formulae (ii) define meromorphic automorphisms of E^{ϵ} which factorise $\mathrm{Ad}(\Psi_i(e^{2\pi \iota u}))$, and then relying on the uniqueness statement of Proposition 4.11.

For $u \notin \overline{D} + \epsilon a - \eta \mathbb{N}^{\times}$, define $\mathcal{G}^{\eta}(u) \in GL(E^{\epsilon})$ by

$$\int_{\mathcal{C}} f(v)\mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv \longrightarrow \int_{\mathcal{C}} \frac{\Gamma^{\eta}(u - v - \epsilon a)}{\Gamma^{\eta}(u - v + \epsilon a)} f(v)\mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv$$

Note that this is well–defined by (i), Lemma 6.5 and the fact that D is non–congruent. Since the right–hand side is a meromorphic function of $u \in \mathbb{C}$, and lies in E^{ϵ} for $u \notin \overline{D} + \epsilon a - \eta \mathbb{N}^{\times}$, it lies in E^{ϵ} for all u.

It is clear that $\mathcal{G}^{\eta}(u)$ are invertible and holomorphic for $\mathrm{Re}(\pm u) \gg 0$, and that $\mathcal{G}^{+}(u)^{-1}\mathcal{G}^{-}(u) = \mathrm{Ad}(\Psi_i(e^{2\pi \iota u}))\big|_{E^{\epsilon}}$. Set

$$\Xi(u) = \mathcal{G}^{+}(u + 1)\mathcal{G}^{+}(u)^{-1} = \mathcal{G}^{-}(u + 1)\mathcal{G}^{-}(u)^{-1}$$

By inspection, $\Xi(u)^{\pm 1}$ is given by

$$\int_{\mathcal{C}} f(v)\mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv \longrightarrow \int_{\mathcal{C}} \left(\frac{u - v + \epsilon a}{u - v - \epsilon a} \right)^{\pm 1} f(v)\mathcal{X}_j^{\epsilon}(e^{2\pi \iota v}) dv$$

for any $u \notin \overline{D} \pm \epsilon a$. Since the poles of $\Xi(u)^{\pm 1}$ are contained in $\overline{D} \pm \epsilon a$, and those of $\mathrm{Ad}(\xi_i(u))_{E^{\epsilon}}$ are contained in Π which is non–congruent, it follows from our assumptions on D and Proposition 4.11 that $\Xi(u) = (\mathrm{Ad}(\xi_i(u))\big|_{E^{\epsilon}}$, and therefore that $\mathcal{G}^{\eta}(u) = (\mathrm{Ad}(g_i^{\eta}(u))\big|_{E^{\epsilon}}$. $\square$

6.7. **Proof of (Y3).** It suffices to prove $(\mathcal{Y}3)$ for u, v large enough. Let $i, j \in \mathbf{I}$, and set $a = \hbar d_i a_{ij}/2$. Since the contour $\mathcal{C}_j^{\pm}$ defining $x_j^{\pm}(v)$ in (6.1) is only required to contain the poles of $\xi_j(v)$, and the latter are contained in Π which is non–congruent and stable under shift by $\hbar/2$, we may choose $\mathcal{C}_j^{\pm}$ so that it also satisfies the conditions (1)–(3) of Proposition 6.6. It then follows that

$$\xi_i(u)x_j^{\pm}(v)\xi_i(u)^{-1} = \frac{1}{c_j^{\pm}} \int_{\mathcal{C}_j^{\pm}} \frac{u - v' \pm a}{(v - v')(u - v' \mp a)} g_j^{\pm}(v')^{-1} \mathcal{X}_j^{\pm}\left(e^{2\pi \iota v'}\right) dv'$$

so long as $u \mp a$ and v do not lie within $\mathcal{C}_j$. $(\mathcal{Y}3)$ now follows from the fact that

$$\frac{u - v' \pm a}{(v - v')(u - v' \mp a)} = \frac{u - v \pm a}{u - v \mp a} \frac{1}{v - v'} \mp \frac{2a}{u - v \mp a} \frac{1}{u \mp a - v'}$$

6.8. **Proof of (Y4).** We shall prove that

$$x_{i,r+1}^{\pm}x_{j,s}^{\pm} - x_{i,r}^{\pm}x_{j,s+1}^{\pm} \mp a x_{i,r}^{\pm}x_{j,s}^{\pm} = x_{j,s}^{\pm}x_{i,r+1}^{\pm} - x_{j,s+1}^{\pm}x_{i,r}^{\pm} \pm a x_{j,s}^{\pm}x_{i,r}^{\pm} \quad (6.3)$$

The left–hand side , multiplied by $c_i^{\pm}c_j^{\pm}$ is equal to

$$\frac{1}{\hbar^2} \int_{\mathcal{C}_i^{\pm}} \int_{\mathcal{C}_j^{\pm}} u^r v^s (u - v \mp a) g_i^{\pm}(u)^{-1} \mathcal{X}_i^{\pm}\left(e^{2\pi \iota u}\right) g_j^{\pm}(v)^{-1} \mathcal{X}_j^{\pm}\left(e^{2\pi \iota v}\right) du\, dv$$

Choose $\mathcal{C}_i^{\pm}, \mathcal{C}_j^{\pm}$ so that they satisfy conditions (a)–(c) of Proposition 6.6 with respect to $\xi_j(v)$ and $\xi_i(u)$ respectively, as well as

$$\mathcal{C}_j^{\pm} \cap \left(\overline{D}_i \pm a \mp \mathbb{N}^{\times}\right) = \emptyset = \mathcal{C}_i^{\pm} \cap \left(\overline{D}_j \pm a \mp \mathbb{N}^{\times}\right)$$

where D_i, D_j are the interior domains of $\mathcal{C}_i^{\pm}, \mathcal{C}_j^{\pm}$. Then, the above is equal to

$$\frac{1}{\hbar^2} \int_{\mathcal{C}_i^{\pm}} \int_{\mathcal{C}_j^{\pm}} u^r v^s (u - v \mp a) \frac{\Gamma^{\pm}(v - u \mp a)}{\Gamma^{\pm}(v - u \pm a)} g_i^{\pm}(u)^{-1} g_j^{\pm}(v)^{-1} \mathcal{X}_i^{\pm}\left(e^{2\pi \iota u}\right) \mathcal{X}_j^{\pm}\left(e^{2\pi \iota v}\right) du\, dv$$

Similarly, the right–hand side of (6.3) multiplied by $c_i^{\pm}c_j^{\pm}$ is equal to

$$\frac{1}{\hbar^2} \int_{\mathcal{C}_i^{\pm}} \int_{\mathcal{C}_j^{\pm}} u^r v^s (u - v \pm a) \frac{\Gamma^{\pm}(u - v \mp a)}{\Gamma^{\pm}(u - v \pm a)} g_i^{\pm}(u)^{-1} g_j^{\pm}(v)^{-1} \mathcal{X}_j^{\pm}\left(e^{2\pi \iota v}\right) \mathcal{X}_i^{\pm}\left(e^{2\pi \iota u}\right) du\, dv$$

By $(\mathcal{QL}4)$, this may be rewritten as

$$\frac{1}{\hbar^2} \int_{\mathcal{C}_i^{\pm}} \int_{\mathcal{C}_j^{\pm}} u^r v^s (u - v \pm a) \frac{\Gamma^{\pm}(u - v \mp a)}{\Gamma^{\pm}(u - v \pm a)} \frac{e^{2\pi \iota(v \pm a)} - e^{2\pi \iota u}}{e^{2\pi \iota v} - e^{2\pi \iota(u \pm a)}}$$
$$\cdot g_i^{\pm}(u)^{-1} g_j^{\pm}(v)^{-1} \mathcal{X}_i^{\pm}\left(e^{2\pi \iota u}\right) \mathcal{X}_j^{\pm}\left(e^{2\pi \iota v}\right) du\, dv$$

Indeed, neither of the boundary terms $e^{2\pi \iota v}\left(\mathcal{X}_{j,0}^{\pm}\mathcal{X}_i^{\pm}(e^{2\pi \iota u}) - q_j^{\pm a_{ji}}\mathcal{X}_i^{\pm}(e^{2\pi \iota u})\mathcal{X}_{j,0}^{\pm}\right)$ and $e^{2\pi \iota u}\left(\mathcal{X}_{i,0}^{\pm}\mathcal{X}_j^{\pm}(e^{2\pi \iota v}) - q_j^{\pm a_{ji}}\mathcal{X}_j^{\pm}(e^{2\pi \iota v})\mathcal{X}_{i,0}^{\pm}\right)$ from $(\mathcal{QL}4)$ contribute to the double integral since $(u - v \pm a)/(e^{2\pi \iota v} - e^{2\pi \iota(u \pm a)})$ does not have any poles inside $\mathcal{C}_i^{\pm} \times \mathcal{C}_j^{\pm}$.

The result now follows from the fact that $\Gamma^\mp(x) = \Gamma^\pm(-x)$, and (5.8).

6.9. **Proof of (Y5).** We need to prove that for any $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$ we have

$$[x_{i,r}^+, x_{j,s}^-] = \delta_{ij}\xi_{i,r+s}$$

Choosing the contours $\mathcal{C}_i^+, \mathcal{C}_j^-$ as in 6.8, Proposition 6.6 yields

$$x_{i,r}^+ x_{j,s}^- = \frac{1}{c_i^+ c_j^- \hbar^2} \int_{\mathcal{C}_i^+} \int_{\mathcal{C}_j^-} u^r v^s g_i^+(u)^{-1} \mathcal{X}_i^+\left(e^{2\pi\iota u}\right) g_j^-(v)^{-1} \mathcal{X}_j^-\left(e^{2\pi\iota v}\right) \, du \, dv$$

$$= \frac{1}{c_i^+ c_j^- \hbar^2} \int_{\mathcal{C}_i^+} \int_{\mathcal{C}_j^-} u^r v^s g_i^+(u)^{-1} g_j^-(v)^{-1} \frac{\Gamma^-(v-u-a)}{\Gamma^-(v-u+a)} \mathcal{X}_i^+\left(e^{2\pi\iota u}\right) \mathcal{X}_j^-\left(e^{2\pi\iota v}\right) \, du \, dv$$

Similarly,

$$x_{j,s}^- x_{i,r}^+ = \frac{1}{c_i^+ c_j^- \hbar^2} \int_{\mathcal{C}_i^+} \int_{\mathcal{C}_j^-} u^r v^s g_i^+(u)^{-1} g_j^-(v)^{-1} \frac{\Gamma^+(u-v+a)}{\Gamma^+(u-v-a)} \mathcal{X}_j^-\left(e^{2\pi\iota v}\right) \mathcal{X}_i^+\left(e^{2\pi\iota u}\right) \, du \, dv$$

Since $\Gamma^+(x) = \Gamma^-(-x)$ it follows that

$$[x_{i,r}^+, x_{j,s}^-] = \frac{1}{c_i^+ c_j^- \hbar^2} \int_{\mathcal{C}_i^+} \int_{\mathcal{C}_j^-} u^r v^s g_i^+(u)^{-1} g_j^-(v)^{-1}$$

$$\cdot \frac{\Gamma^+(u-v+a)}{\Gamma^+(u-v-a)} [\mathcal{X}_i^+\left(e^{2\pi\iota u}\right), \mathcal{X}_j^-\left(e^{2\pi\iota v}\right)] \, du \, dv$$

By $(\mathcal{QL}5)$, we get

$$[x_{i,r}^+, x_{j,s}^-] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \frac{1}{c_i^+ c_j^- \hbar^2} \int_{\mathcal{C}_i^+} \int_{\mathcal{C}_j^-} u^r v^s g_i^+(u)^{-1} g_j^-(v)^{-1}$$

$$\cdot \frac{\Gamma^+(u-v+a)}{\Gamma^+(u-v-a)} \frac{\Psi_i\left(e^{2\pi\iota v}\right) - e^{2\pi\iota(v-u)}\Psi_i\left(e^{2\pi\iota u}\right)}{1 - e^{2\pi\iota(v-u)}} \, du \, dv$$

since the contribution of the boundary term $\Psi_{i,0}^-$ to the double integral is zero given that the function $\Gamma^+(u-v+a)$ is holomorphic in $D_i \times D_j$. We may therefore assume that $i = j$.

Note that

$$\frac{\Psi_i\left(e^{2\pi\iota v}\right) - e^{2\pi\iota(v-u)}\Psi_i\left(e^{2\pi\iota u}\right)}{1 - e^{2\pi\iota(v-u)}} = \frac{\Psi_i\left(e^{2\pi\iota v}\right) - \Psi_i\left(e^{2\pi\iota u}\right)}{1 - e^{2\pi\iota(v-u)}} + \Psi_i\left(e^{2\pi\iota u}\right)$$

and that the second summand on the right–hand side does not contribute to the double integral. Choosing the contours $\mathcal{C}_i^\pm$ to be equal to a common contour $\mathcal{C}_i$ which contains all poles of $\xi_i(u)$, we may apply Lemma 5.8 to find

$$[x_{i,r}^+, x_{i,s}^-] = \frac{1}{q_i - q_i^{-1}} \frac{1}{c_i^+ c_i^- \hbar^2} \frac{\Gamma^+(\hbar d_i)}{\Gamma^-(\hbar d_i)} \int_{\mathcal{C}_i} u^{r+s} g_i^+(u)^{-1} g_i^-(u)^{-1} \Psi_i\left(e^{2\pi\iota u}\right) \, du$$

where we used the fact that

$$\lim_{u \to v} \frac{u-v}{1 - e^{2\pi\iota(v-u)}} = \frac{1}{2\pi\iota}$$

Using (5.9), (5.5) and (4.9), we then get

$$[x_{i,r}^+, x_{i,s}^-] = \frac{1}{2\pi\iota\hbar} \int_{\mathcal{C}_i} u^{r+s} \xi_i(u)\, du = \xi_{i,r+s}$$

6.10. **Proof that $\daleth_\Pi \circ \Gamma_\Pi = 1$.** Let $V \in \mathcal{O}_{\mathrm{int}}^{\Pi}(Y_\hbar(\mathfrak{g}))$. By (5.7), $x_i^\pm(u)$ acts on $\daleth_\Pi \circ \Gamma_\Pi(V)$ as

$$\frac{1}{c_i^\pm} \int_{\mathcal{C}_i^\pm} \frac{1}{u-v} g_i^\pm(v)^{-1} \mathcal{X}_i^\pm(v)\, dv$$

$$= \int_{\mathcal{C}_i^\pm} \frac{1}{u-v} g_i^\pm(v)^{-1} \left(\oint_{\mathcal{C}'_i^\pm} \frac{1}{1 - e^{-2\pi\iota(v-v')}} g_i^\pm(v') x_i^\pm(v')\, dv' \right) dv$$

where u lies outside of $\mathcal{C}_i^\pm$, and $\mathcal{C}_i^\pm$ lies outside of $\mathcal{C}'_i^\pm + \mathbb{Z}$. Assuming that $\mathcal{C}_i^\pm$ contains $\mathcal{C}'_i^\pm$ and none of its $\mathbb{Z}^\times$–translates, and integrating in v first yields

$$\frac{1}{2\pi\iota} \int_{\mathcal{C}'_i^\pm} \frac{1}{u-v'} x_i^\pm(v')\, dv' = x_i^\pm(u)$$

where the equality follows from the fact that $\mathcal{C}'_i^\pm$ contains all the poles of $x_i^\pm(u)$.

6.11. **Proof that $\Gamma_\Pi \circ \daleth_\Pi = 1$.** Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}^{\Omega}(U_q(L\mathfrak{g}))$. Then, $\mathcal{X}_i^\pm(z)$ acts on $\Gamma_\Pi \circ \daleth_\Pi(\mathcal{V})$ as

$$c_i^\pm \oint_{\mathcal{C}_i^\pm} \frac{z}{z - e^{2\pi\iota v}} g_i^\pm(v) x_i^\pm(v)\, dv$$

$$= \oint_{\mathcal{C}_i^\pm} \frac{z}{z - e^{2\pi\iota v}} g_i^\pm(v) \int_{\mathcal{C}'_i^\pm} \frac{1}{v - v'} g_i^\pm(v')^{-1} \mathcal{X}_i^\pm(e^{2\pi\iota v'})\, dv'\, dv$$

where z lies outside of $\exp(2\pi\iota\mathcal{C}_i^\pm)$, and $\mathcal{C}_i^\pm$ lies outside of $\mathcal{C}'_i^\pm$. Assuming that $\mathcal{C}_i^\pm$ contains $\mathcal{C}'_i^\pm$, and integrating in v first yields

$$\int_{\mathcal{C}'_i^\pm} \frac{z}{z - e^{2\pi\iota v'}} \mathcal{X}_i^\pm(e^{2\pi\iota v'})\, dv' = \int_{\widetilde{\mathcal{C}}'_i^\pm} \frac{z}{z-w} \mathcal{X}_i^\pm(w)\, \frac{dw}{w}$$

where $\widetilde{\mathcal{C}}'_i^\pm = \exp(2\pi\iota\mathcal{C}'_i^\pm)$ contains all the poles of $\mathcal{X}_i^\pm$. Since the one–form $dw/(z-w)w$ is regular on $\mathbb{P}^1 \setminus \{z, 0\}$, and $\mathcal{X}_i^\pm(0) = 0$, the above is equal to $\mathcal{X}_i^\pm(z)$.

7. q–CHARACTERS

7.1. **q–characters for $U_q(L\mathfrak{g})$.** The q–characters of finite–dimensional representations of the quantum loop algebra $U_q(L\mathfrak{g})$ of a semisimple Lie algebra were introduced by Frenkel–Reshetikhin in [24], and later extended to the case of a symmetrisable Kac–Moody algebra by Hernandez in [31, 32]. We follow the presentation in [31, 32].

Let $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$. Given a collection $\gamma = (\gamma_{i,\pm m}^{\pm})_{i\in\mathbf{I}, m\in\mathbb{N}}$ of complex numbers, and $\mu \in \mathfrak{h}^*$ such that $q_i^{\pm\mu(\alpha_i^\vee)} = \gamma_{i,0}^{\pm}$, define $\mathcal{V}[\mu, \gamma]$ as follows

$$\mathcal{V}[\mu, \gamma] = \{v \in \mathcal{V} \,|\, K_h v = q^{\mu(h)}v, \text{ and}$$
$$\exists p > 0 \text{ such that } (\Psi_{i,\pm m}^{\pm} - \gamma_{i,\pm m}^{\pm})^p v = 0 \,\forall i \in \mathbf{I}, m \in \mathbb{N}\}$$

Proposition. *For a given pair (μ, γ), set*

$$\gamma_i(z)^{\pm} = \sum_{m\geq 0} \gamma_{i,\pm m}^{\pm} z^{\mp m}$$

Then, there exists $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ such that $\mathcal{V}[\mu, \gamma] \neq 0$ if, and only if

(i) *There exist monic polynomials $\{\mathcal{Q}_i(w), \mathcal{R}_i(w)\}$ such that $\mathcal{Q}_i(0) \neq 0$, $\mathcal{R}_i(0) \neq 0$ and for every $i \in \mathbf{I}$ we have*

$$\gamma_i(z)^{\pm} = q_i^{-\deg(\mathcal{Q}_i)+\deg(\mathcal{R}_i)} \frac{\mathcal{Q}_i(q_i^2 z)}{\mathcal{Q}_i(z)} \frac{\mathcal{R}_i(z)}{\mathcal{R}_i(q_i^2 z)} \qquad (7.1)$$

(ii) *$\mu \leq \lambda$ for some $\lambda \in P_+$.*

7.2. Consider the collection of variables $\{Y_{i,a}\}_{i\in\mathbf{I}, a\in\mathbb{C}^\times}$, and let $\mathcal{A}$ be the abelian group consisting of elements $e(\mu) \cdot M$, where $\mu \in \mathfrak{h}^*$ and M is a Laurent monomial in the variables $\{Y_{i,a}\}$ such that

$$\mu(\alpha_i^\vee) = \sum_{a\in\mathbb{C}^\times} \text{degree of } Y_{i,a} \text{ in } M$$

The group operation on $\mathcal{A}$ is $e(\mu) \cdot M e(\mu') \cdot M' = e(\mu + \mu') \cdot MM'$, with $e(0) \cdot 1$ being the unit element.

Let $\mathcal{Y}$ be the group algebra of $\mathcal{A}$ over $\mathbb{Z}$, completed in the following sense. An element of $\mathcal{Y}$ is a formal (possibly infinite) linear combination χ of elements of $\mathcal{A}$ with $\mathbb{Z}$ coefficients, satisfying the following condition: there exist $\lambda_1, \cdots, \lambda_r \in P_+$ such that if $e(\mu) \cdot M$ appears in χ with non–zero coefficient, then $\mu \leq \lambda_j$ for some $j \in \{1, \ldots, r\}$.

7.3. To each (μ, γ) of the form given by Proposition 7.1, we associate a monomial $M(\mu, \gamma) \in \mathcal{Y}$ as follows. Let $\{\mathcal{Q}_{i,\gamma}, \mathcal{R}_{i,\gamma}\}$ be a collection of polynomials associated to γ so that (7.1) holds. If

$$\mathcal{Q}_{i,\gamma} = \prod_j (w - a_j^{(i)}) \qquad \text{and} \qquad \mathcal{R}_{i,\gamma} = \prod_k (w - b_k^{(i)})$$

set

$$M(\mu, \gamma) = e(\mu) \cdot \prod_{i\in\mathbf{I}} \left(\prod_j Y_{i,a_j^{(i)}}\right) \left(\prod_k Y_{i,b_k^{(i)}}\right)^{-1}$$

The q–character of $\mathcal{V} \in \mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ is defined by

$$\chi_q^U(\mathcal{V}) = \sum_{(\mu,\gamma)} \dim(\mathcal{V}[\mu, \gamma]) M(\mu, \gamma) \in \mathcal{Y} \qquad (7.2)$$

It gives rise to a homomorphism of abelian groups

$$\chi_q^U : K\left(\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))\right) \to \mathcal{Y}$$

Remark. When $\mathfrak{g}$ is a simple Lie algebra, the tensor structure on $\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))$ induces a ring structure on $K\left(\mathcal{O}_{\mathrm{int}}(U_q(L\mathfrak{g}))\right)$. In this case, it is proved in [24] that χ_q^U is an injective ring homomorphism.

7.4. **q–characters for $Y_\hbar(\mathfrak{g})$.** q–characters for the Yangian $Y_\hbar(\mathfrak{g})$ of a semisimple Lie algebra were introduced by Knight in [37]. The treatment in [31, 32] readily carries over to the Yangian, and allows to extend their definition to the case of a symmetrisable Kac–Moody algebra.

Given a collection $\beta = (\beta_{i,r})_{i\in\mathbf{I},r\in\mathbb{N}}$ of complex numbers, $\mu \in \mathfrak{h}^*$ such that $d_i\mu(\alpha_i^\vee) = \beta_{i,0}$, and $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$, define

$$V[\mu,\beta] = \{v \in V : h.v = \mu(h)v, \text{ and}$$
$$\exists p > 0 \text{ such that } (\xi_{i,r} - \beta_{i,r})^p v = 0 \,\forall\, i \in \mathbf{I}, r \in \mathbb{N}\}$$

Theorem. [37] *Given a pair (μ, β) as above, set*

$$\beta_i(u) = 1 + \hbar \sum_{r\geq 0} \beta_{i,r} u^{-r-1} \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$$

Then, there exists $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ such that $V[\mu,\beta] \neq 0$ if, and only if

(i) *There exist monic polynomials $\{Q_i, R_i\}_{i\in\mathbf{I}}$ such that for every $i \in \mathbf{I}$ we have*

$$\beta_i(u) = \frac{Q_i(u + \hbar d_i)}{Q_i(u)} \frac{R_i(u)}{R_i(u + \hbar d_i)} \tag{7.3}$$

(ii) *$\mu \leq \lambda$ for some $\lambda \in P_+$.*

7.5. Define a commutative ring $\mathcal{X}$ analogous to $\mathcal{Y}$, over the collection of variables $\{X_{i,a}\}_{i\in\mathbf{I},a\in\mathbb{C}}$. Namely, consider the abelian group of monomials $e(\mu) \cdot m$ as in §7.2, except that m is now a Laurent monomial in $\{X_{i,a}\}$. Then, $\mathcal{X}$ is the group algebra of this abelian group over $\mathbb{Z}$, completed in the same way as in §7.2.

7.6. To a pair (μ, β) of the form given by Theorem 7.4, we associate a monomial $m(\mu, \beta) \in \mathcal{X}$ as follows. Let $\{Q_i, R_i\}$ be a set of monic polynomials associated to β so that (7.3) holds. If

$$Q_i = \prod_j (u - a_j^{(i)}) \qquad \text{and} \qquad R_i = \prod_k (u - b_k^{(i)})$$

set

$$m(\mu, \beta) = e(\mu) \cdot \prod_{i\in\mathbf{I}} \left(\prod_j X_{i,a_j^{(i)}}\right) \left(\prod_k X_{i,b_k^{(i)}}\right)^{-1}$$

The q–character of $V \in \mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))$ is now defined as

$$\chi_q^Y(V) := \sum_{(\mu,\beta)} \dim(V[\mu,\beta]) m(\mu,\beta) \in \mathcal{X} \qquad (7.4)$$

and gives rise to a homomorphism of abelian groups

$$\chi_q^Y : K\left(\mathcal{O}_{\mathrm{int}}(Y_\hbar(\mathfrak{g}))\right) \to \mathcal{X}$$

For a semisimple Lie algebra $\mathfrak{g}$, χ_q^Y is an injective ring homomorphism [37].

7.7.   Let $\mathcal{X}_\Pi \subset \mathcal{X}$ be the subring generated by $\mathfrak{h}^*$ and the variables $\{X_{i,a}\}_{i\in\mathbf{I},a\in\Pi}$. By Theorem 3.8, χ_q^Y restricts to a homomorphism

$$\chi_q^Y : K(\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))) \to \mathcal{X}_\Pi$$

Similarly, let $\mathcal{Y}_\Omega \subset \mathcal{Y}$ be the subring generated by $\mathfrak{h}^*$ and $\{Y_{i,\alpha}\}_{i\in\mathbf{I},\alpha\in\Omega}$. Again by Theorem 3.8, χ_q^U restricts to a homomorphism

$$\chi_q^U : K(\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))) \to \mathcal{Y}_\Omega$$

Consider the isomorphism $e_\Pi : \mathcal{X}_\Pi \to \mathcal{Y}_\Omega$ which is identity on $\mathfrak{h}^*$ and maps $X_{i,a}$ to $Y_{i,e^{2\pi\iota a}}$. The following result follows from the definition of the functor Γ and the computation in 4.9.

Proposition. *The following diagram is commutative*

$$
\begin{array}{ccc}
K(\mathcal{O}_{\mathrm{int}}^\Pi(Y_\hbar(\mathfrak{g}))) & \xrightarrow{\ (\Gamma_\Pi)_*\ } & K\left(\mathcal{O}_{\mathrm{int}}^\Omega(U_q(L\mathfrak{g}))\right) \\
\chi_q^Y \downarrow & & \downarrow \chi_q^U \\
\mathcal{X}_\Pi & \xrightarrow[\ e_\Pi\ ]{} & \mathcal{Y}_\Omega
\end{array}
$$

In particular, for a simple Lie algebra $\mathfrak{g}$, $(\Gamma_\Pi)_*$ is an isomorphism of commutative rings.

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